On the matrices of capacitance and inductance coefficients for lossless homogeneous multiconductor transmission lines

A. Reibiger

Technische Universität Dresden, Fakultät Elektrotechnik, Professur für Theoretische Elektrotechnik, Mommsenstr. 13, 01062 Dresden, Germany

Abstract. Using results from the theory of planar fields and complex functions it is shown that for lossless homogeneous multiconductor transmission lines the computation of the matrices of capacitance and inductance coefficients can be reduced to the solution of some special Dirichlet boundary value problems. Additionally, a function theoretic proof is given for the relationship $\mathbf{LC} = \epsilon\mu\mathbf{E}$ between these matrices.

1 Introduction

Besides of rare exceptions the standard textbook representations of transmission line equations for two-conductor transmission lines start with a finite LC-ladder network as a model for a lossless transmission line. The system of transmission line equations is then developed by means of some questionable limit processes.

Let Λ denote a multiconductor transmission line consisting of $n+1$ perfect conductors embedded in a perfectly insulating, homogeneous dielectric with permittivity ϵ and permeability μ. The propagation of TEM waves on Λ can be described in terms of voltages and currents by means of the transmission line equations $\frac{\partial \mathbf{u}}{\partial z} = -\mathbf{L}\frac{\partial \mathbf{i}}{\partial t}$, $\frac{\partial \mathbf{i}}{\partial z} = -\mathbf{C}\frac{\partial \mathbf{u}}{\partial t}$ where $\mathbf{L}$ and $\mathbf{C}$ are the $n \times n$ matrices of inductance and capacitance coefficients, resp. Under the assumption that the matrices $\mathbf{L}$ and $\mathbf{C}$ exist it is possible to derive the transmission line equations from the system of Maxwell's equations. For the special case of two-conductor transmission lines, i.e. for the case $n = 1$, such a proof was known to Heaviside (1971) at least in 1883. But it may be that this fact had been known to him even already since 1876 (Heaviside, 1970). As a corollary of such a proof it follows that these matrices obey the relationship $\mathbf{LC} = \epsilon\mu\mathbf{E}$, a fact seemingly first observed by Wagner (1914). Since the columns of the matrix $\mathbf{C}$ can be computed by solving n special Dirichlet boundary value problems this

formula is very useful. It allows the computation of $\mathbf{L}$ by means of $\mathbf{L} = \epsilon\mu\mathbf{C}^{-1}$.

The weak point in the just mentioned argumentation is the assumption that the matrix $\mathbf{L}$ of the inductance coefficients exists. Elementary approaches for the computation of $\mathbf{L}$, e.g. application of Biot-Savart's law etc., do not work because (i) the surface current density distribution is unknown and (ii) it is impossible to guarantee that the magnetic field does not penetrate the conductors although they are assumed to be perfectly conducting. But more sophisticated methods are also difficult to apply.

By means of the theory of planar fields and complex function theory (Henrici, 1974; Lawrentjew and Schabat, 1967) we present a direct proof of the identity $\mathbf{LC} = \epsilon\mu\mathbf{E}$. Starting with the above mentioned solutions of n special Dirichlet boundary value problems we construct first of all n complex basis potentials for the set of all transversal electrostatic fields on Λ. Since the cross section of Λ is a multiply connected domain it is necessary for this construction to introduce n suitable branch cuts. The complex potential for any transversal electrostatic field on Λ can then be represented as a linear combination of these complex basis potentials. In the next step we show that a simple multiplication of these complex basis potentials with the complex constant -j delivers the complex basis potentials for transversal stationary magnetic fields on Λ such that the complex potential of any transversal stationary magnetic field on Λ can be represented as a linear combination of the complex basis potentials obtained in this way. Complex contour integrals enclosing the cross sections of the conductors of Λ deliver then the elements of the matrices $\mathbf{C}$ and $\mathbf{\Gamma} := \mathbf{L}^{-1}$. A simple coefficient comparision delivers the identity in question. The n branch cuts necessary for a unique definition of the complex potentials are essentially used for the evaluation of the complex contour integrals. Last but not least, a classical result of potential theory for two-dimensional domains guarantees the existence of all our complex basis potentials if the conductor boundaries are sufficiently smooth.
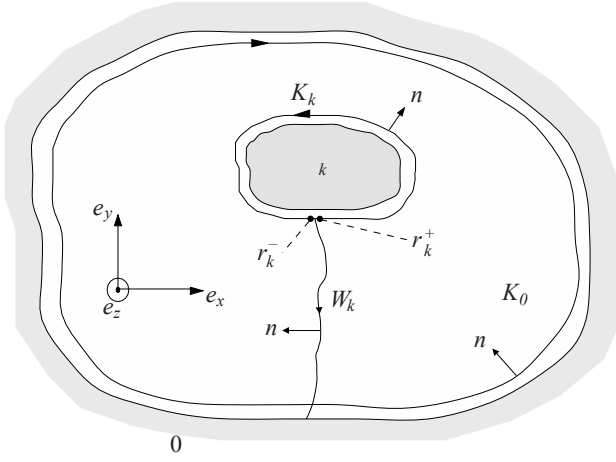
Correspondence to: A. Reibiger
(reibiger@iee.et.tu-dresden.de)

Fig. 1. Graphical representation of the cross section of Λ.

2 Matrices of capacitance and inductance coefficients for lossless homogeneous multiconductor transmission lines

We consider a multiconductor transmission line Λ with the following properties:

Λ consists of n perfectly conducting cylindric inner conductors with constant cross sections which are parallel to each other and of infinite length. These conductors are enclosed by an perfectly conducting cylindric outer conductor which is also of infinite length.

We assume that the electromagnetic fields do not penetrate these conductors. Therefore, it is only necessary to admit surface current densities and surface charge densities, denoted in the following by $\mathcal{S}$ and σ, resp. As it is standard we denote the electric field strength by $\mathcal{E}$, the displacement density by $\mathcal{D}$, the magnetic field strength by $\mathcal{H}$, and the magnetic induction by $\mathcal{B}$.

The space between the conductors is filled with a perfect insulating dielectric with permittivity ϵ and permeability μ.

The conductors of Λ are denoted by the natural numbers $0, 1,..., n$. The boundary surface between the conductors and the insulator is denoted by A, and $v := 1/\sqrt{\epsilon\mu}$ denotes the light velocity in the insulator material.

Figure 1 shows a cross section of Λ. For simplicity we have shown in Fig. 1 only the outer conductor, denoted by 0, and that inner conductor which is denoted by the number k. In figure 1 there are also shown the integration paths K_0, K_k, and W_k and their normal vectors. For $k = 1, ..., n$ the path W_k starts at conductor k and ends at conductor 0. The paths W_k ($k = 1, ..., n$) are pairwise disjoint and and each path W_k does not intersect or touch one of the conductors denoted by $l \in \{1, ..., n\} \setminus \{k\}$. Subsequently, these paths and normal vectors are used for the definition of voltages and currents. As reference directions for the currents in the conductors $k = 1, ..., n$ we use the positive orientation of the z-axis. Because of the right-hand rule convention included implicitly in Ampère's integral law we have to use the

oppositely oriented paths $(-W_k)$ for the computation of the magnetic fluxes per unit length trough the W_k ($k = 1, ..., n$) by means of the function theoretic methods mentioned in the introduction.

With respect to the coordinate system sketched in Fig. 1 by means of the basic vectors e_x, e_y, e_z we introduce the projection Ξ defined by the assignment $(x, y, z) \mapsto (x, y)$ and the embedding Θ defined by $(x, y) \mapsto (x, y, 0)$.

For simplicity of representation we assume additionally that the surfaces of the conductors are sufficiently smooth and that all functions considered in the following are at least two times continously differentiable.

Next, we turn towards the determination of the matrices of capacitance and inductance coefficients. These matrices are characteristics of the electrostatic and the stationary magnetic fields on Λ, resp.

An ordered triple $(\mathcal{E}, \mathcal{D}, \sigma)$ is an *electrostatic field* on Λ if it is a solution of $\text{rot}\mathcal{E} = 0$, $\text{div}\mathcal{D} = 0$, $\mathcal{D} = \epsilon\mathcal{E}$, $\sigma = n\cdot\mathcal{D}|A$, and $n \times \mathcal{E}|A = 0$.

An ordered triple $(\mathcal{H}, \mathcal{B}, \mathcal{S})$ is an *stationary magnetic field* on Λ if it is a solution of $\text{rot}\mathcal{H} = 0$, $\text{div}\mathcal{B} = 0$, $\mathcal{B} = \mu\mathcal{H}$, $\mathcal{S} = n \times \mathcal{H}|A$, and $n \cdot \mathcal{B}|A = 0$.

For the computation of the matrices of capacitance and inductance coefficients we are only interested in z-independent transversal solutions of these equations since we need these matrices for a description of TEM waves. Therefore we always assume that the the conditions $\mathcal{E} = (E_x, E_y, 0)$, $\mathcal{D} = (D_x, D_y, 0)$, $\mathcal{H} = (H_x, H_y, 0)$, $\mathcal{B} = (B_x, B_y, 0)$, and $\mathcal{S} = (0, 0, S_z)$ are fulfilled. Thus, nontransversal electrostatic and stationary magnetic fields on Λ as discussed in Nähring (2002) are excluded.

By means of $U_k := \int_{W_k} \mathcal{E} \cdot dr$ ($k = 1, ..., n$) we assign to each transversal electrostatic field on Λ the *associated voltage matrix* $\mathbf{U} := {}^t(U_1, ..., U_n)$. And by means of $I_k := \int_{K_k} \mathcal{S} \cdot e_z ds = \int_{K_k} (n \times \mathcal{H}) \cdot e_z ds = -\int_{K_k} \mathcal{H} \cdot (n \times e_z) ds = \int_{K_k} \mathcal{H} \cdot (e_z \times n) ds$, i.e. $I_k := \int_{K_k} \mathcal{H} \cdot dr$ ($k = 1, ..., n$), we assign to each transversal stationary magnetic field on Λ the *associated current matrix* $\mathbf{I} := {}^t(I_1, ..., I_n)$.

Now we are able to set up and to prove the following

Theorem *There are symmetric, positive definite $n \times n$-matrices $\mathbf{C}$, $\boldsymbol{\Gamma}$ and $\mathbf{L}$ with*

$$\mathbf{C} = \epsilon\mu\boldsymbol{\Gamma}, \ \mathbf{L} = \boldsymbol{\Gamma}^{-1}$$

such that for each transversal static electric field $(\mathcal{E}, \mathcal{D}, \sigma)$ on Λ and its associated voltage matrix $\mathbf{U}$, and for each transversal stationary magnetic field $(\mathcal{H}, \mathcal{B}, \mathcal{S})$ on Λ and its associated current matrix $\mathbf{I}$ the conditions

$$\oint_{K_k} \mathcal{D} \cdot n \, ds = \mathbf{C}_{k\bullet}\mathbf{U}$$

and

$$\oint_{-W_k} \mathcal{B} \cdot n \, ds = \mathbf{L}_{k\bullet}\mathbf{I}$$

are fulfilled for each $k = 1, ..., n$ where $\mathbf{C}_{k\bullet}$ and $\mathbf{L}_{k\bullet}$ denotes the k-th row of the matrices $\mathbf{C}$ and $\mathbf{L}$, resp.

Proof: In the following G denotes the complete image of the space between the conductors of Λ under the projection Ξ.

Let $(\mathcal{E}, \mathcal{D}, \sigma)$ be a transversal electrostatic field on Λ. Because the conductor surfaces are always equipotential surfaces of the $\mathcal{E}$-field the line integrals $\oint_{K_k} \mathcal{E} \cdot t \, \mathrm{d}s$ are vanishing for all $k = 0, 1, ..., n$. Therefore the potential function of $\Xi \circ \mathcal{E} \circ \Theta$ is globally defined on G and the computation of the corresponding electrostatic field can be reduced to that one of a Dirichlet boundary value problem.

In a first step we consider on G the n special Dirichlet boundary value problems $\Delta_2 \, \varphi_l = 0$, $\varphi_l|K_0 = 0$, $\varphi_l|K_l = \delta_{kl}$ $(k, l = 1, ..., n)$ where Δ_2 denotes the differential operator defined for each C^2-function φ by means of $\Delta_2 \, \varphi := \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2}$ and δ_{kl} denotes the so called Kronecker symbol defined for all $k, l = 1, ..., n$ by $\delta_{kk} := 1$ and $\delta_{kl} := 0$ $(k \neq l)$.

The boundary conditions determining the functions φ_l $(l = 1, ..., n)$ imply that the φ_l are a system of linearly independent functions. Obviously, their linear combination $\varphi = \sum_l \varphi_l U_l$ delivers for each family of voltages U_l $(l = 1, ..., n)$ a solution of the boundary value problem given by $\Delta_2 \, \varphi = 0$, $\varphi|K_0 = 0$, $\varphi|K_l = U_l$ $(l = 1, ..., n)$.

From standard uniqueness arguments it follows then that the φ_l $(l = 1, ..., n)$ are not only linearly independent but that they are a basis for the set of the solutions of all these boundary value problems, too.

By means of suitable branch cuts, as such one can be used the curves W_l $(l = 1, ..., n)$, we associate to each function φ_l a conjugate function ψ_l. The functions ψ_l are then defined on a simply connected subdomain of G which we denote by G^*.

An ordered triple $(\mathcal{E}, \mathcal{D}, \sigma)$ is a transversal electrostatic field on Λ if and only if there exists a family $(U_l)_{l=1,...,n}$ such that the associated planar field $\Xi \circ \mathcal{E} \circ \Theta$ can be represented in either of the two forms $\Xi \circ \mathcal{E} \circ \Theta(r) = -\mathrm{grad}_2 \varphi(r)$ $(r \in G)$, $\Xi \circ \mathcal{E} \circ \Theta(r) = -\mathrm{rot}_1 \psi(r)$ $(r \in G^*)$ where φ and ψ are the real and imaginary part of a complex potential $\varphi|G^* + \mathrm{j}\psi$ defined by $\varphi := \sum_l \varphi_l U_l$, $\psi := \sum_l \psi_l U_l$ and the nonstandard differential operators grad_2 and rot_1 are defined for each scalar C^1-field φ by means of $\mathrm{grad}_2 \varphi := (\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y})$ and $\mathrm{rot}_1 \varphi := (\frac{\partial \varphi}{\partial y}, -\frac{\partial \varphi}{\partial x})$, respectively.

Let $(\mathcal{H}, \mathcal{B}, \mathcal{S})$ be a transversal stationary magnetic field on Λ. Because there do not exist magnetic monopols the line integrals $\oint_{K_k} \mathcal{B} \cdot n \, \mathrm{d}s$ are vanishing for all $k = 0, 1, ..., n$. Therefore the stream function of $\Xi \circ \mathcal{B} \circ \Theta$ is globally defined on G. Together with the fact that for $l = 1, ..., n$ each equipotential line of $-\mathrm{grad}_2 \varphi_l$ is a field line of $-\mathrm{rot}_1 \varphi_l$, and vice versa, this observation motivates for a complex potential $\vartheta + \mathrm{j}\chi|G^*$ of $\Xi \circ \mathcal{B} \circ \Theta$ the ansatz $\vartheta := +\sum_l \psi_l \Phi_l$, $\chi := -\sum_l \varphi_l \Phi_l$ where $(\Phi_l)_{l=1,...,n}$ is a family of flux values per unit length. And indeed, an ordered triple $(\mathcal{H}, \mathcal{B}, \mathcal{S})$ is a transversal stationary magnetic field if and only if there exists a family $(\Phi_l)_{l=1,...,n}$ such that the associated planar field $\Xi \circ \mathcal{B} \circ \Theta$ can be represented by means of the just defined complex potential in either of the two forms $\Xi \circ \mathcal{B} \circ \Theta(r) = -\mathrm{grad}_2 \vartheta(r)$ $(r \in G^*)$, $\Xi \circ \mathcal{B} \circ \Theta(r) = -\mathrm{rot}_1 \chi(r)$ $(r \in G)$. Together with the limit values $\Psi_{kl}^+ := \psi_l(r_k^+)$, $\Psi_{kl}^- := \psi_l(r_k^-)$ of the functions ψ_l $(l = 1, ..., n)$ on the left and right edges of the corresponding branch cuts (cf. fig. 1) it follows

for the charge Q_k per unit length on conductor k and for the current I_k in this conductor the relationship

$$Q_k = \oint_{K_k} \mathcal{D} \cdot n \, \mathrm{d}s = \sum_l \epsilon(\Psi_{kl}^+ - \Psi_{kl}^-)U_l,$$

$$I_k = \oint_{K_k} \mathcal{H} \cdot t \, \mathrm{d}s = \sum_l \mu^{-1}(\Psi_{kl}^+ - \Psi_{kl}^-)\Phi_l,$$

respectively. And with $C_{kl} := \epsilon(\Psi_{kl}^+ - \Psi_{kl}^-)$, $\Gamma_{kl} := \mu^{-1}(\Psi_{kl}^+ - \Psi_{kl}^-)$, $\mathbf{C} := (C_{kl})_{k,l=1,...,n}$, $\mathbf{\Gamma} := (\Gamma_{kl})_{k,l=1,...,n}$ there results $C_{kl} = \epsilon\mu\Gamma_{kl}$ $(k, l = 1, ..., n)$, i.e.

$$\mathbf{C} = \epsilon\mu\mathbf{\Gamma} = \epsilon\mu\mathbf{L}^{-1},$$

$$\mathbf{LC} = \epsilon\mu\mathbf{E}.$$

What are to be proved.

To understand the physical meaning of the parameters Φ_{kl} and Γ_{kl} let us consider for $k \in \{1, ..., n\}$ a family $(\Phi_l)_{l=1,...,n}$ defined by $\Phi_k \neq 0$ and $\Phi_l = 0$ $(l \in \{1, ..., n\} \setminus \{k\})$. If we denote the start point of W_k by r_k^s and the end point of W_k by r_k^e then

$$\int_{-W_k} \mathcal{B} \cdot n \, \mathrm{d}s = -\int_{-W_k} \mathrm{rot}_1 \chi \cdot n \, \mathrm{d}s$$
$$= \int_{W_k} \mathrm{rot}_1 \chi \cdot n \, \mathrm{d}s$$
$$= \chi(r_k^e) - \chi(r_k^s)$$
$$= (\varphi_k(r_k^s) - \varphi_k(r_k^e))\Phi_k$$
$$= \varphi_k(r_k^s)\Phi_k$$
$$= \Phi_k.$$

Thus Φ_k is equal to the flux per unit length through $(-W_k)$, and $\mathbf{I} = \mathbf{\Gamma}_{\bullet k}\Phi_k$ is the matrix of that conductor currents for which the fluxes per unit length trough the $(-W_l)$ for $l \in \{1, ..., n\} \setminus \{k\}$ are vanishing and that one trough $(-W_k)$ is equal to Φ_k where $\mathbf{\Gamma}_{\bullet k}$ denotes the kth column of $\mathbf{\Gamma}$.

It can be shown (Jost, 2002) that for sufficiently smooth conductor surfaces all the functions exist and that they fulfill the assumptions necessary to applicate results of of the theory of planar fields.

The symmetry of the matrices $\mathbf{C}$ and $\mathbf{L}$ can be proved by means of a variant of the Green's integral theorems specialized for planar fields. The positive definiteness of these matrices follows from the positive definiteness of the energy stored in the electric and magnetic field of Λ.

3 Concluding remarks

Based on Wagner (1914); Abraham and Föppl (1917); Adler et al. (1960); Helm (1975), we have presented in Reibiger (2002) a thorough field theoretic approach for the description of the propagation of TEM waves on lossless multiconductor transmission lines by means of the system of transmission line equations. Together with the remarks guaranteeing the

existence of the matrices of capacitance and inductance co-
efficients for such transmission lines from that approach it
follows also the existence of TEM waves on lossless multi-
conductor lines.

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