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Statistical multipole formulations for shielding problems

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Abstract. A multipole-based method is presented for modelling an electromagnetic field with small statistical variations inside an arbitrary enclosure. The accurate computation of the statistics of the field components from the statistical moments of the multipole amplitudes is demonstrated for two- and three-dimensional examples. To obtain the statistics of quantities which depend non-linearly on the field components, higher-order statistical moments of the latter are required.

1 Introduction

Recent research in the field of statistical electromagnetics in general and statistical EMC in particular seems to focus on extremely non-deterministic fields as observed in a reverberation chamber. Much progress has been achieved in this area over the last decades, leading to very general statistical power distributions largely independent of the chamber geometry (Holland, 1999). Relatively little work has been done on the topic of small statistical variations of an otherwise known electromagnetic field. A related example relevant to EMC has been presented in (Ajayi et al., 2008).

In this paper we discuss the case of small statistical variations in various parameters of a shielding problem, e.g. the angle of incidence of a plane wave impinging on a shield. The varying parameters are given by the first few statistical moments of their distributions. From these the statistical moments of the amplitudes of a spherical-multipole expansion are derived. The statistics of the multipole amplitudes then characterizes the statistical properties of the electromagnetic field not only in a single point but in a spherical region around the arbitrarily chosen center of the expansion. In practice, this spherical region can be part of the enclosed volume of any shielding structure. For non-canonical problems, where the calculation of the multipole amplitudes cannot be achieved analytically, data from numerical simulations or measurements are needed to estimate the statistical moments of the multipole amplitudes.

2 Multipole expansion

Consider an arbitrary shielding structure as depicted in Fig. 1.

The electromagnetic field in any linear, homogeneous, and source-free spherical sub-domain of the enclosure can be expanded by means of a spherical-multipole expansion:

$$\boldsymbol{E} = \sum_{n=1}^{\infty}\sum_{m=-n}^{n} A_{n,m}\boldsymbol{N}_{n,m} + \frac{Z}{j}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} B_{n,m}\boldsymbol{M}_{n,m} \qquad (1)$$

$$\boldsymbol{H} = \frac{j}{Z}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} A_{n,m}\boldsymbol{M}_{n,m} + \sum_{n=1}^{\infty}\sum_{m=-n}^{n} B_{n,m}\boldsymbol{N}_{n,m}. \qquad (2)$$

The coefficients $A_{n,m}$ and $B_{n,m}$ are referred to as the electric and magnetic multipole amplitudes, respectively, $Z = \sqrt{\mu/\varepsilon}$ is the wave impedance in the spherical sub-domain and $\boldsymbol{M}_{n,m}$, $\boldsymbol{N}_{n,m}$ represent the spherical-multipole functions defined by

$$\boldsymbol{M}_{n,m} = (\boldsymbol{r}\times\nabla)\, j_n(\kappa r) Y_{n,m}(\vartheta,\varphi) \qquad (3)$$

$$= j_n(\kappa r)\boldsymbol{m}_{n,m}(\vartheta,\varphi) \qquad (4)$$

$$\boldsymbol{N}_{n,m} = \left[\frac{1}{\kappa}\nabla\times(\boldsymbol{r}\times\nabla)\right] j_n(\kappa r) Y_{n,m} \qquad (5)$$

$$= -\frac{j_n(\kappa r)}{\kappa r} n(n+1) Y_{n,m}\hat{\boldsymbol{r}} - \frac{1}{\kappa r}\frac{d}{dr}[r j_n(\kappa r)]\boldsymbol{n}_{n,m}. \qquad (6)$$

Here, $j_n(\kappa r)$ denote spherical Bessel functions of the 1st kind, $\hat{\boldsymbol{r}}$ is the radial unit vector, and the vector wave functions are found as:

$$\boldsymbol{m}_{n,m}(\vartheta,\varphi) = -\frac{1}{\sin\vartheta}\frac{\partial Y_{n,m}}{\partial\varphi}\hat{\boldsymbol{\vartheta}} + \frac{\partial Y_{n,m}}{\partial\vartheta}\hat{\boldsymbol{\varphi}} \qquad (7)$$

$$\boldsymbol{n}_{n,m}(\vartheta,\varphi) = +\frac{\partial Y_{n,m}}{\partial\vartheta}\hat{\boldsymbol{\vartheta}} + \frac{1}{\sin\vartheta}\frac{\partial Y_{n,m}}{\partial\varphi}\hat{\boldsymbol{\varphi}}. \qquad (8)$$

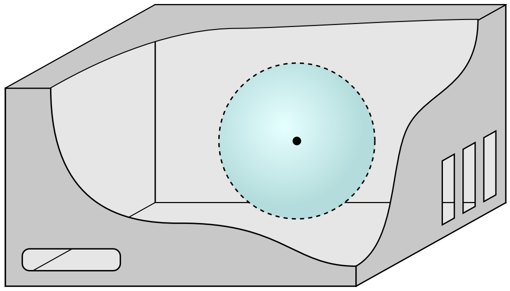

Fig. 1. Spherical domain inside an arbitrary shielding enclosure.

The normalized surface-spherical harmonics are given by

$$Y_{n,m}(\vartheta,\varphi) = \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}}\,P_n^m(\cos\vartheta)e^{jm\varphi} \tag{9}$$

with P_n^m being associated Legendre functions of the 1st kind.

For two-dimensional problems, a cylindrical-multipole expansion in plane-polar coordinates (R,φ) consisting of ordinary Bessel-functions J_n in R and harmonic functions in φ can be applied. For the $\mathrm{TM_z}$-case this takes the following form (Klinkenbusch, 2005):

$$E_z(R,\varphi) = \sum_{n=-\infty}^{\infty} a_n J_n(\kappa R)e^{jn\varphi} \tag{10}$$

$$H_R(R,\varphi) = \frac{j}{Z}\frac{1}{\kappa R}\frac{\partial}{\partial\varphi}E_z(R,\varphi) \tag{11}$$

$$H_\varphi(R,\varphi) = -\frac{j}{Z}\frac{1}{\kappa}\frac{\partial}{\partial R}E_z(R,\varphi). \tag{12}$$

In any of the described cases, the complete information about the field is contained in the complex-valued multipole amplitudes ($A_{n,m}$, $B_{n,m}$; a_n). Consequently, the statistics of an electromagnetic field inside the enclosure can be reduced to the statistics of the corresponding multipole amplitudes.

3 Statistical formulation

To describe a statistically varying electromagnetic field we assume its multipole amplitudes to be random variables. For a spherical-multipole expansion truncated at $n = n_{\max}$ there are $n_{\max}(n_{\max}+2)$ index pairs (n,m). For each (n,m) four real random variables need to be considered:

$$\mathrm{Re}\{A_{n,m}\}, \mathrm{Im}\{A_{n,m}\}, \mathrm{Re}\{B_{n,m}\}, \mathrm{Im}\{B_{n,m}\}. \tag{13}$$

We introduce the multipole-amplitude random vector $\boldsymbol{V}$ consisting of all multipole amplitudes considered. It thus has the length $i_{\max} = 4n_{\max}(n_{\max}+2)$.

This random vector $\boldsymbol{V}$ will be described by its first few moments, particularly by its expectation value η_V and its covariance matrix C_V (Papoulis, 2008).

3.1 Computation of field statistics

Once the multipole-amplitude random vector $\boldsymbol{V}$ is known in terms of its statistical moments, the field statistics at any points in the domain, where the multipole expansion is valid, can be easily calculated. To this end, we first construct a new field-component random vector $\boldsymbol{W}$ consisting of all real and imaginary parts of all field components at any desired point in the domain. Because of the linearity of the multipole expansion (Eqs. 1–2) the field-component random vector $\boldsymbol{W} = g(\boldsymbol{V})$ depends linearly on the multipole-amplitude random vector $\boldsymbol{V}$, hence the statistical moments of $\boldsymbol{W}$ are easily obtained from the statistical moments of $\boldsymbol{V}$. For the first two moments (expectation value and covariance matrix) this leads to:

$$\eta_{W_k} = g_k(\eta_V) \tag{14}$$

$$C_{W_k,W_l} = \sum_i \sum_j C_{V_i,V_j} \left.\frac{\partial g_k}{\partial v_i}\right|_{\eta_V} \left.\frac{\partial g_l}{\partial v_j}\right|_{\eta_V}. \tag{15}$$

Note that for this linear relation n-th order moments of $\boldsymbol{W}$ only depend on n-th order moments of $\boldsymbol{V}$.

This means that the complete field statistics of the whole domain, where the multipole expansion is valid, is contained in the moments of $\boldsymbol{V}$. Usually, the first two moments are sufficient, for arbitrary accuracy higher order moments like skew, curtosis, etc. – as well as their correlations – may have to be considered.

3.2 Compactness of description

The proposed approach of a statistical multipole expansion is a very efficient and systematic way of modelling an electromagnetic field with small variations.

As an example consider the field in a cubic volume of edge length 1.6 m centered around the point of origin at a frequency of 750 MHz (as used in the first examples of Sect. 4). In this case the multipole expansion can be truncated at $n_{\max} \approx 20$ for sufficient accuracy, which leads to a random vector of length $i_{\max} \approx 1760$. On the other hand, to describe the statistics of the field components at each point in space, at least six real random variables would be required per position, i.e. the quadrature components of either the electric or the magnetic field. At a reasonable spatial resolution of at least ten points per wavelength this amounts to a random vector of length $i_{\max} \approx 384000$.

The impact of this difference is even larger when considering that, while the length of the vector of expectation values η is $i_{\max}$, the number of entries in the correlation matrix C is $i_{\max}^2$.

A similar result is obtained for a cylindrical expansion of a square domain of 1.6 m×1.6 m. At 750 MHz the statistical multipole expansion leads to $i_{\max} \approx 194$, the pointwise description of the field components' statistics to $i_{\max} \approx 3200$.

3.3 Sources of statistical variations

To test this new approach we start with a simple example, that is, the electromagnetic field of a homogeneous plane wave where we add a small statistical variation using a single normally distributed real random variable U.

The spherical multipole expansion's (Eqs. 1–2) amplitudes of a plane wave are known (Klinkenbusch, 1996):

$$A_{n,m} = E_0 4\pi j^{n+1} \frac{(-1)^m}{n(n+1)} \left[\boldsymbol{n}_{n,-m}(\vartheta,\varphi)\cdot\hat{\boldsymbol{\xi}} \right]_{\vartheta_0,\varphi_0} \tag{16}$$

$$B_{n,m} = \frac{E_0}{Z} 4\pi j^{n+1} \frac{(-1)^m}{n(n+1)} \left[\boldsymbol{m}_{n,-m}(\vartheta,\varphi)\cdot\hat{\boldsymbol{\xi}} \right]_{\vartheta_0,\varphi_0}. \tag{17}$$

Here, E_0 and $\hat{\boldsymbol{\xi}}$ are amplitude and polarization of the plane wave, while (ϑ_0,φ_0) denote its angle of incidence.

For the two-dimensional $\mathrm{TM_z}$-case, the multipole amplitudes of a plane wave of amplitude E_0 and angle of incidence φ_0 are (Klinkenbusch, 2005):

$$a_n = E_0 j^n e^{-jn\varphi_0}. \tag{18}$$

First we will vary the phase. This can be easily done by multiplying the known amplitudes by a phase factor containing the random variable U as follows:

$$A'_{n,m} = A_{n,m} e^{jU} \tag{19}$$

$$B'_{n,m} = B_{n,m} e^{jU} \tag{20}$$

$$a'_n = a_n e^{jU}. \tag{21}$$

A practically more interesting case is the variation of the angle of incidence. This will be realized by replacing the angle φ_0 in Eqs. (16–18) with a random variable U as described above.

With these relations between the given random variable U and the multipole-amplitude random vector $\boldsymbol{V} = g(U)$, the moments of $\boldsymbol{V}$ can be calculated. Since U is a single real random variable (not a random vector), the expectation values and covariance matrix of $\boldsymbol{V}$ are as follows:

$$\eta_{V_i} = g_i(\eta_U) + \sum_{n=2}^{\infty} \frac{\mu_n(U)}{n!} g_i^{(n)}(\eta_U) \tag{22}$$

$$C_{V_i,V_j} = \sum_{n=2}^{\infty} \frac{\mu_n(U)}{n!} \left(\sum_{k=1}^{n-1} \binom{n}{k} g_i^{(k)}(\eta_U) g_j^{(n-k)}(\eta_U) \right) - \sum_{n=4}^{\infty} \left(\sum_{k=2}^{n-2} \frac{\mu_k(U)\mu_{n-k}(U)}{k!(n-k)!} g_i^{(k)}(\eta_U) g_j^{(n-k)}(\eta_U) \right). \tag{23}$$

Here $\mu_n(U)$ denotes the n-th central moment of U and $g_i^{(k)}$ is the k-th derivative of g_i with respect to its argument.

In this paper we choose U to be normally distributed with zero mean, hence all even moments of U are known in terms of its standard deviation σ_U (Abramowitz, 1972) and all odd moments are zero (see Table 1). For the following calculations the moments of U are considered up to and including eighth order.

Table 1. Central moments of a normally distributed random variable (Abramowitz, 1972).

n	1	2	3	4	5	6	7	8	...
μ_n	0	σ^2	0	$3\sigma^4$	0	$15\sigma^6$	0	$105\sigma^8$	...

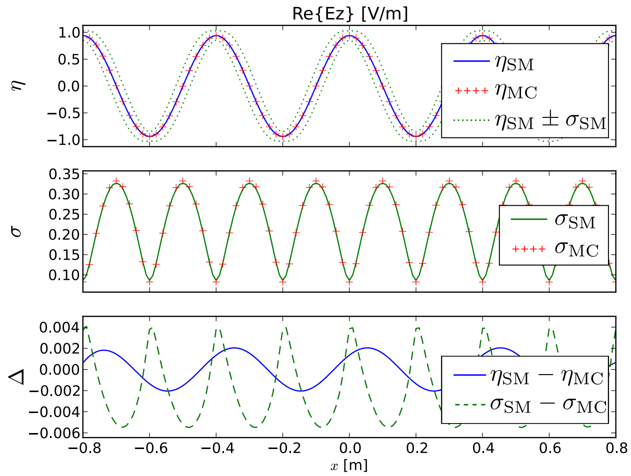

Fig. 2. Expectation value η (upper plot) and standard deviation σ (central plot) of Re$\{E_z\}$ according to statistical multipole calculation (SM) and Monte Carlo simulation (MC) for a plane wave with varying phase ($\sigma_\psi = 20°$). The lower plot displays the differences between statistical multipole calculation and Monte Carlo simulation.

4 Results

4.1 Plane wave

Figure 2 shows the results for a plane wave with a varying phase ψ of standard deviation $\sigma_\psi = 20°$ at frequency $f = 750$ MHz. The upper diagram represents the expectation value η of the real part of E_z and includes the corresponding standard deviation σ_{SM}. The central diagram shows the standard deviation σ_{SM} alone. Both are calculated by the aforementioned method and – for reference – by a Monte-Carlo simulation (Index $_{\mathrm{MC}}$). The absolute differences between the spherical-multipole calculations are also shown in the lower diagram.

As expected, for a varying phase the standard deviation is largest at the inflection points of the sine curve and minimal at the extreme values. The accordance with Monte Carlo simulations is excellent.

Figure 3 shows the results for a plane wave along the main axis, where the angle of incidence φ_0 is varied according to a standard deviation of $\sigma_{\varphi_0} = 5°$.

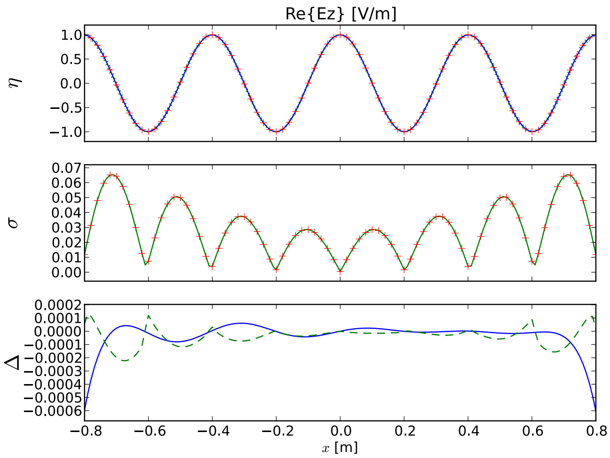

Fig. 3. Expectation value and standard deviation of $\mathrm{Re}\{E_z\}$ according to statistical multipole calculation and Monte Carlo simulation for a plane wave with varying incident angle ($\sigma_{\varphi_0} = 5°$). For a legend see Fig. 2.

Again, the influence of the angle of incidence on the field is largest at the sine curve's inflexion points and minimal at the extreme values, and the agreement with the Monte Carlo simulations is excellent.

4.2 Slitted cylinder

To test our approach with a simple shielding geometry we have used the geometry shown in Fig. 4 representing a thin slitted PEC-cylinder with radius R_S and aperture angle β, illuminated by a plane wave polarized in the z-direction and incident at φ_0. For the deterministic solution of this problem see (Klinkenbusch, 2005).

Figure 5 shows the expectation value of the imaginary part of E_z in the x-y-plane. The shape of the cylinder with $R_0 = 0.5$ m and $\beta = 40°$ is displayed as a black line, the angle of incidence is varying around its expectation value $\eta_{\varphi_0} = 135°$ with standard deviation $\sigma_{\varphi_0} = 5°$. The frequency of the incident field is 750 MHz.

Figure 6 shows in detail the corresponding expectation values, standard deviations and their comparison to Monte Carlo simulations (as in Figs. 2–3) along the x-axis (the dashed line in Fig. 5).

Compared to the previous examples the field structure is much more complex and the standard deviation reaches much larger values. However the multipole expansion again shows excellent correspondence with the Monte Carlo simulation particularly inside the shield.

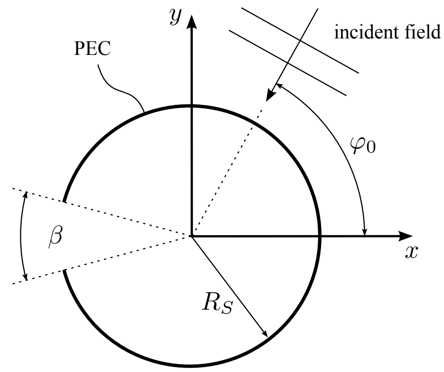

Fig. 4. Geometry of the slitted PEC cylinder in the x-y-plane.

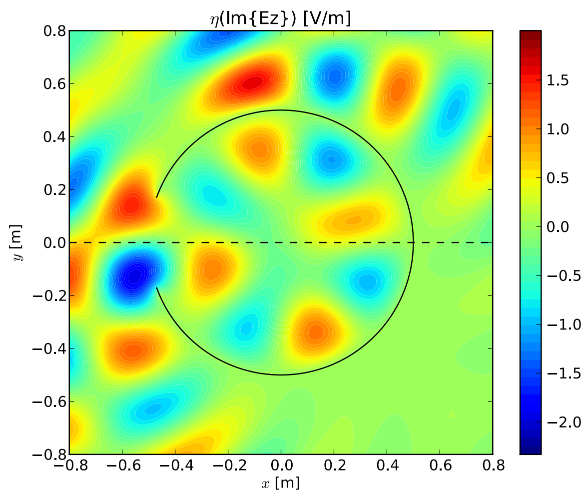

Fig. 5. Expectation value of $\mathrm{Im}\{E_z\}$ according to statistical multipole calculation for a slitted cylinder illuminated by a plane wave with varying angle of incidence ($\sigma_{\varphi_0} = 5°$).

4.3 Shielding effectiveness

The electromagnetic shielding effectiveness has been shown to be an adequate quantity for the characterization of a shield for the high-frequency case (Klinkenbusch, 1996). It is defined as:

$$SE_{em} = 10 \log_{10} \frac{2}{\frac{|\boldsymbol{E}^{\mathrm{sh}}|^2}{|\boldsymbol{E}^{\mathrm{un}}|^2} + \frac{|\boldsymbol{H}^{\mathrm{sh}}|^2}{|\boldsymbol{H}^{\mathrm{un}}|^2}}. \tag{24}$$

The quantities marked with $^{\mathrm{sh}}$ are for the shielded case, while those marked $^{\mathrm{un}}$ are for the unshielded case, i.e. the case where no shield is present at all. For the example in Sect. 4.2 the corresponding unshielded fields $|\boldsymbol{E}^{un}|$ and $|\boldsymbol{H}^{un}|$ are those of an undisturbed plane wave and thus are constant. The fields for the shielded case are those of Sect. 4.2 described by the random vector $\boldsymbol{W}$. The random variable for

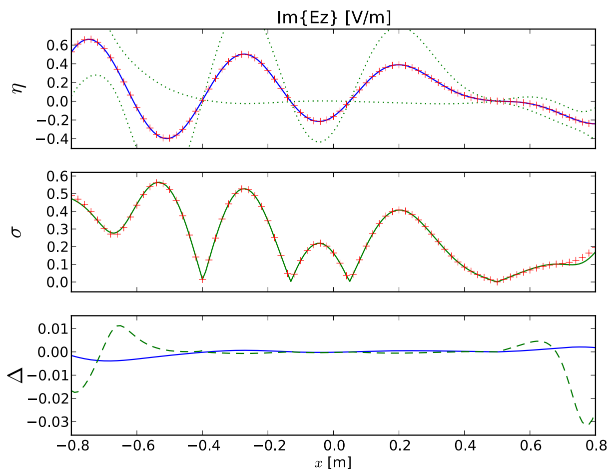

Fig. 6. Expectation value and standard deviation of $\mathrm{Im}\{E_z\}$ according to statistical multipole calculation and Monte Carlo simulation for a slitted cylinder illuminated by a plane wave with varying angle of incidence ($\sigma_{\varphi_0} = 5°$). For a legend see Fig. 2.

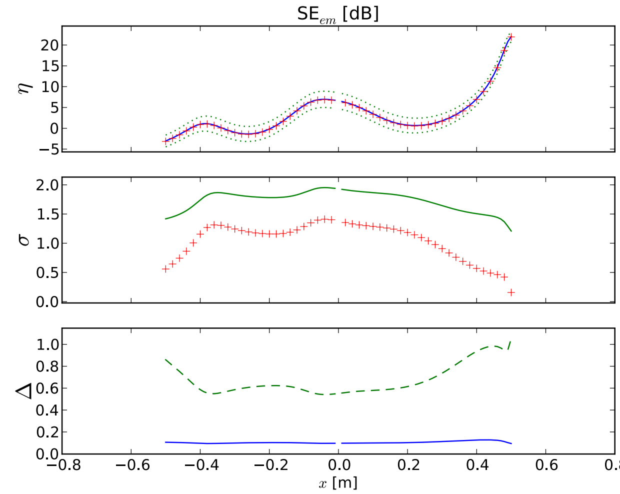

Fig. 8. Expectation value and standard deviation of SE_{em} according to statistical multipole calculation and Monte Carlo simulation for a slitted cylinder illuminated by a plane wave with varying angle of incidence ($\sigma_{\varphi_0} = 5°$). For a legend see Fig. 2.

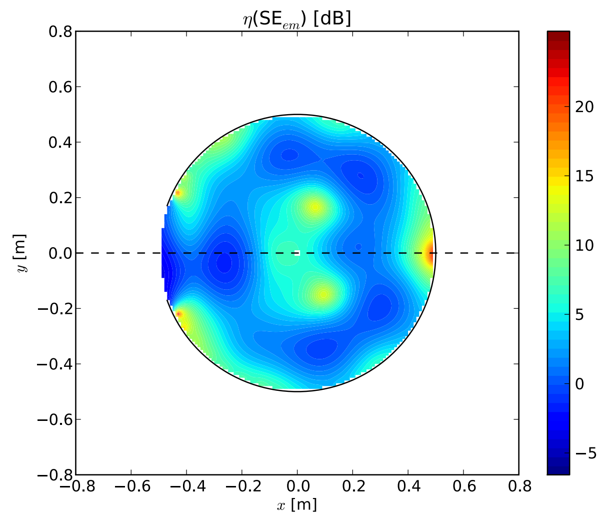

Fig. 7. Expectation value of SE_{em} according to statistical multipole calculation for a slitted cylinder illuminated by a plane wave with varying angle of incidence ($\sigma_{\varphi_0} = 5°$).

the shielding effectiveness will be called X. To compare the results with those ones of the simple cases described before we consider the expectation value and the standard deviation of the shielding effectiveness. These are calculated from the relations

$$\eta_X = g(\eta_{\boldsymbol{W}}) + \sum_i \sum_j \frac{C_{\boldsymbol{W}_i,\boldsymbol{W}_j}}{2} \left.\frac{\partial^2 g}{\partial w_i \partial w_j}\right|_{\eta_{\boldsymbol{W}}} + \ldots \tag{25}$$

$$\sigma_X^2 = \sum_i \sum_j C_{\boldsymbol{W}_i,\boldsymbol{W}_j} \left.\frac{\partial g}{\partial w_i}\right|_{\eta_{\boldsymbol{W}}} \left.\frac{\partial g}{\partial w_j}\right|_{\eta_{\boldsymbol{W}}} + \ldots \tag{26}$$

Figure 7 shows the expectation value of SE_{em} in the x–y-plane. Figure 8 shows in detail the corresponding expectation values, standard deviations and their comparison to Monte Carlo simulations along the x-axis. All parameters are identical to those in Figs. 5–6.

While the error of the expectation value is still quite small, the standard deviation derived from the multipole evaluation differs significantly from the Monte Carlo simulation. This can be explained by Eqs. (25–26). The moments of X depend on all moments of $\boldsymbol{W}$, but unlike Eqs. (22–23), where all moments of the normally distributed random variable U were known, in this case only the expectation values and covariance matrix of $\boldsymbol{W}$ have been taken into account for this calculation. Obviously that is not sufficient for a non-linear relation like Eq. (24).

This clearly shows that for the statistics of quantities that depend in a non-linear fashion on the field components, also higher order moments of the multipole amplitudes have to be taken into account.

5 Conclusions

For calculating the statistical moments of the electromagnetic field in case of small variations of various parameters, the statistical-multipole method yields similar results as Monte Carlo simulations with the advantage that the number of quantities to represent the field and its statistics is greatly reduced. However, modeling the multipole-amplitude random vector in terms of only first and second order statistical moments has been shown to be insufficient for the computation of the moments of quantities, which nonlinearly depend on the multipole amplitudes like the electromagnetic shielding effectiveness. In such cases higher-order statistical moments of the statistical multipole expansion have to be considered. However, the basic advantage of representing the field by a minimum number of parameters is still preserved.

Further research includes the application of the method to more general, that is, non-canonical shielding problems, and to characterize the shielding effect by the first few terms of the corresponding multipole expansion and its statistical moments.

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