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Numerical quadrature for the approximation of singular oscillating integrals appearing in boundary integral equations

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Abstract. Boundary Integral Equation formulations can be used to describe electromagnetic shielding problems. Yet, this approach frequently leads to integrals which contain a singularity and an oscillating part. Those integrals are difficult to handle when integrated naivly using standard integration techniques, and in some cases even a very high number of integration nodes will not lead to precise results.

We present a method for the numerical quadrature of an integral with a logarithmic singularity and a cosine oscillator: a modified Filon-Lobatto quadrature for the oscillating parts and an integral transformation based on the error function for the singularity. Since this integral can be solved analytically, we are in a position to verify the results of our investigations, with a focus on precision and computation time.

1 Introduction

We investigate the field properties of a rectangular bar of constant conductivity κ_i and permeability μ_i. The space outside the bar has the arbitrary constant permeability μ_a and conductivity $\kappa_\mathrm{a}=0$. The vector $\boldsymbol{r}_q = x_q\boldsymbol{e}_x + y_q\boldsymbol{e}_y$ is pointing at the lower left corner of the bar, the dimensions of it being a and b, as shown in Fig. 1. We name the cross-section of the bar Ω and its contour $C=\partial\Omega$.

An exciting loop consists of two thin conductors, which carry the currents $i(t)$ and $-i(t)$, with $i(t)=\Re\{I e^{j\omega t}\}$, I being the phasor describing the complex current . They are located at

$$\boldsymbol{r}_\mathrm{e} = \pm x_e \boldsymbol{e}_x .$$

The angular frequency ω is considered to be low enough so that displacement currents can be neglected: $|\boldsymbol{J}| \gg |\frac{\partial}{\partial t}\boldsymbol{D}|$, where $\boldsymbol{J}$ is the current density inside the bar and $\boldsymbol{D}$ the electric flux.

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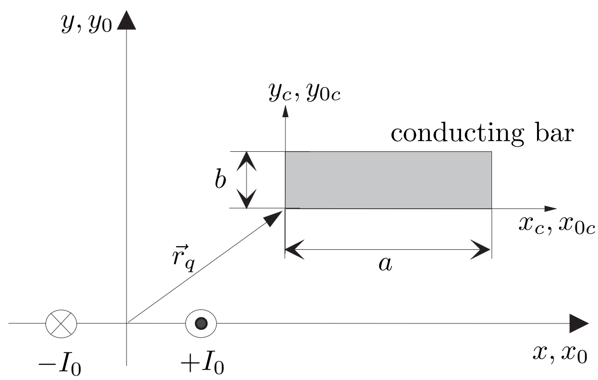

Fig. 1. Screening bar and exciting loop.

The z-directed dimensions of both bar and loop are supposed to be infinite.

2 Differential equations for the vector potential

As all exciting currents are oscillating at one single frequency, all fields will show the same time dependency and can be expressed by their complex phasors.

For a complex vector potential $\boldsymbol{A}$ as $\boldsymbol{B}=\mathrm{curl}\boldsymbol{A}$, with the magnetic flux $\boldsymbol{B}$, we find that for points inside the bar an additional term has to be added to the vector potential. The additional term can be expressed as the time domain integral of the gradient of a complex electrical scalar potential $\varphi_\mathrm{i}(z)$ by

$$\int (\mathrm{grad}\,\varphi_\mathrm{i})\mathrm{d}t = C\boldsymbol{e}_z,$$

with an unknown complex constant C. The vector potential $\boldsymbol{A}_\mathrm{i}$ inside the bar can be redefined by using the Buchholz convention

$$\boldsymbol{A}_\mathrm{i}^* = \boldsymbol{A}_\mathrm{i} + \int (\mathrm{grad}\,\varphi_\mathrm{i})\mathrm{d}t = \boldsymbol{A}_\mathrm{i} + C,$$

from which the fields can be computed with

$$\boldsymbol{B}_\mathrm{i} = \mathrm{curl}\,\boldsymbol{A}_\mathrm{i}^* \quad \text{and} \quad \boldsymbol{E}_\mathrm{i} = -j\omega\boldsymbol{A}_\mathrm{i}^*.$$

From Maxwell's equations we find that the modified vector potential inside the bar is governed by Helmholtz' equation, which for complex fields takes the form

$$\Delta\boldsymbol{A}_\mathrm{i}^* - j\omega\mu_\mathrm{i}\kappa_\mathrm{i}\boldsymbol{A}_\mathrm{i}^* = 0. \tag{1}$$

Outside the bar the equation

$$\Delta\boldsymbol{A}_\mathrm{a} = -\mu_\mathrm{a}\boldsymbol{J}_e, \tag{2}$$

is valid for the vector potential $\boldsymbol{A}_\mathrm{a}$. The exciting current density $\boldsymbol{J}_e$ can be expressed as

$$\boldsymbol{J}_e = \{\delta_1(x-x_e) - \delta_1(x+x_e)\}\delta_1(y)\cdot I\boldsymbol{e}_z.$$

The symbol $\delta_1(x-x_o)$ denotes the one-dimensional Dirac distrubution; accordingly, δ_2 and δ_3 are the two- and three-dimensional Dirac distribution.

Since we allow only z-directed exciting currents, all vector potentials will also be exclusively z-directed. They can be described by their z-components A_az and A_iz^*, respectively. The boundary conditions for A_az and A_iz^* are

$$A_\mathrm{az} = A_\mathrm{iz}^* - C \text{ and } \frac{1}{\mu_\mathrm{a}}\boldsymbol{n}_o\cdot\mathrm{grad}A_\mathrm{az} = \frac{1}{\mu_\mathrm{i}}\boldsymbol{n}_o\cdot\mathrm{grad}A_\mathrm{iz}^*,$$

where $\boldsymbol{n}_o$ is a unit vector normal to the boundary.

3 Derivation of boundary integral equation (BIE)

In Green's second theorem

$$\iiint\limits_{\tau}(\Phi\,\Delta\Psi - \Psi\,\Delta\Phi)\mathrm{d}\tau = \oiint\limits_{F=\partial\tau}(\Phi\,\mathrm{grad}\Psi - \Psi\,\mathrm{grad}\Phi)\cdot\boldsymbol{n}_o\,\mathrm{d}F,$$

we insert the single component of the vector potential A_a as Ψ and a kernel function K satisfying $\Delta K = -\delta_3$ as Φ. As a result, we get an integral representation valid for A_a:

$$A_\mathrm{a} = -\oiint\limits_{F=\partial\tau}(A_\mathrm{a}\,\mathrm{grad}K - K\,\mathrm{grad}A_\mathrm{a})\cdot\boldsymbol{n}_o\,\mathrm{d}F + \mu_\mathrm{a}I\int_{-\infty}^{+\infty}K\Big|_{\substack{x_o=\pm x_e\\ y_o=0}}\mathrm{d}z_o.$$

The integrand of the surface integral at the right hand side of the above equation does not depend on z. Hence the z-directed integration is only affecting the kernel function K, for which the elementary Kernel function

$$K = \frac{1}{4\pi r},$$

is inserted. Thereby we can define a new kernel function G_{20} as

$$G_{20} = -\frac{1}{2\pi}\ln\left(\frac{\rho}{\rho_o}\right),$$

with $\rho = \sqrt{(x-x_o)^2+(y-y_o)^2}$ and $\rho_o = \mathrm{const.}$ While we are aware that the above integral is divergent, it can be solved by performing a suitable renormalization.

The function G_{20} satisfies $\Delta G_{20} = -\delta_2$, as is shown in Ehrich et al. (2000).

If we use the new kernel and define the influence of the exciting loop as

$$A_\mathrm{e} := \mu_\mathrm{a}IG_{20}\Big|_{\substack{x_o=\pm x_e\\ y_o=0}},$$

we can write the integral representation for A_a as

$$A_\mathrm{a} = -\oint\limits_{C=\partial\Omega}(A_\mathrm{a}\,\mathrm{grad}G_{20} - G_{20}\,\mathrm{grad}A_\mathrm{a})\cdot\boldsymbol{n}_o\,\mathrm{d}s + A_\mathrm{e}.$$

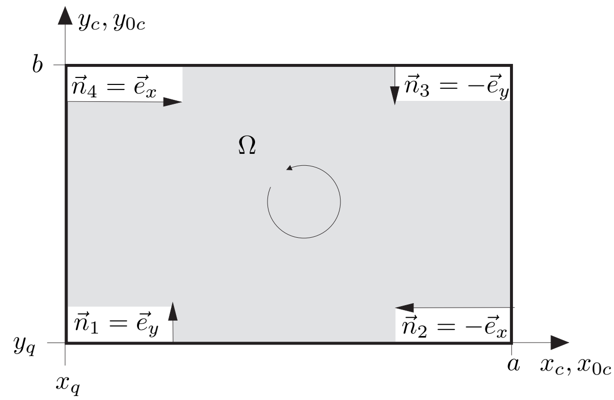

Fig. 2. Normals on boundary sections.

We use the cross-section of the bar as integration domain Ω with its facet normals defined as shown in Fig. 2.

By taking into account the boundary conditions discussed above, we get an integral representation for the vector potential outside the bar:

$$A_\mathrm{a} = -\oint\limits_{C=\partial\Omega}\Big[(A_\mathrm{i}^* - C)\,\mathrm{grad}\,G_{20} - G_{20}\frac{\mu_\mathrm{a}}{\mu_\mathrm{i}}\,\mathrm{grad}A_\mathrm{i}^*\Big]\cdot\boldsymbol{n}_o\,\mathrm{d}s + A_\mathrm{e}. \tag{3}$$

It can be shown that

$$\oint\limits_{C=\partial\Omega}(C\,\mathrm{grad}\,G_{20})\cdot\boldsymbol{n}_o\,\mathrm{d}s = 0,$$

so we can omit this term in Eq. (3).

In a last step we take the normal derivative of A_a on each of the four contour sections of Ω and move the observation point to the very same section, i.e.

$$\begin{array}{ll}(1)\ y = y_q, & (3)\ y = y_q + b,\\ (2)\ x = x_q + a, & (4)\ x = x_q.\end{array}$$

As a result, we get four independent integral equations

$$\frac{\mu_\mathrm{a}}{\mu_\mathrm{i}}\boldsymbol{n}_o \cdot \operatorname{grad} A_\mathrm{i}^* = \boldsymbol{n}_o \cdot \operatorname{grad}\{ \oint_{C=\partial\Omega} (A_\mathrm{i}^* \operatorname{grad} G_{20} - G_{20}\frac{\mu_\mathrm{a}}{\mu_\mathrm{i}} \operatorname{grad} A_\mathrm{i}^*)\, \boldsymbol{n}_o \, \mathrm{d}s + A_\mathrm{e}\}\Big|_{\substack{x=x_q \vee x=x_q+a,y\\ x,y=y_q \vee y=y_q+b}} . \tag{4}$$

We find a solution to Eq. (1) by separation, using a product of functions depending only on one coordinate for the descrition of $\boldsymbol{A}_\mathrm{i}^*$.

$$\begin{aligned} A_\mathrm{i}^*(x_c, y_c) = \sum_{n=1}^{\infty} \big[&v_{1n}\cosh(\beta_n(b-y_c)) \\ &+v_{3n}\cosh(\beta_n y_c)\big]\cos(\alpha_n x_c) \\ &+\big[v_{2n}\cosh(\tilde\beta_n(a-x_c)) \\ &+v_{4n}\cosh(\tilde\beta_n x_c)\big]\cos(\tilde\alpha_n y_c), \end{aligned} \tag{5}$$

$$\alpha_n = \frac{n\pi}{a},\ \beta_n = \sqrt{\alpha_n^2 + j\omega\kappa\mu_\mathrm{i}},$$
$$\tilde\alpha_n = \frac{n\pi}{b},\ \tilde\beta_n = \sqrt{\tilde\alpha_n^2 + j\omega\kappa\mu_\mathrm{i}}.$$

Then, the unknown constants $v_{in},\ i = 1, ..., 4$ will have to be determined.

We can now insert the result for A_i^* into Eq. (4). Using the orthogonality of the cosine functions

$$\int_0^{2\pi} \cos(n\xi)\cos(p\xi)\mathrm{d}\xi = \begin{cases} 0 & n \neq p \\ \pi & n = p \neq 0 \\ 2\pi & n = p = 0 \end{cases},$$

we can isolate one coefficient v_{im} on the left hand side of Eq. (4). Expanding the right hand side into a series of cosine functions by multiplication with $\cos(\alpha_m x_c)$ (or $\cos(\tilde\alpha_m y_c)$, respectively) and computing its integral along one of the contour sections we get

$$\frac{\mu_\mathrm{a}}{\mu_\mathrm{i}} v_{im} d_{im} = \sum_{i=1}^{4}\sum_{n=1}^{\infty}[\gamma_{inm} v_{in}] + b_m, \quad m \int \mathbb{N}.$$

If we take only the first N series elements into account, this equation can be written as a matrix equation

$$\boldsymbol{\Gamma}\boldsymbol{v} = \boldsymbol{b}, \tag{6}$$

with matrix $\boldsymbol{\Gamma}$ and vectors $\boldsymbol{v}$, $\boldsymbol{b}$ of finite dimensions. Equation (6) represents a system of linear equations from which the $4N$ unknown coefficients v_{in} can be computed.

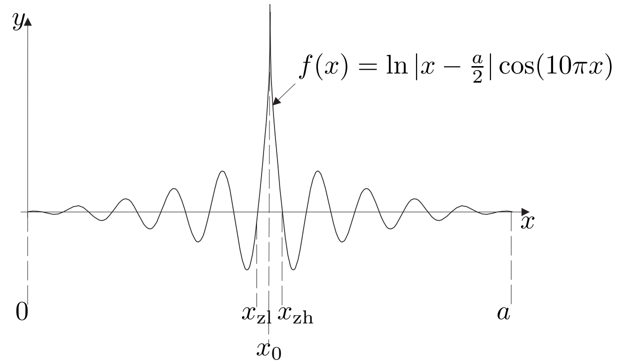

Fig. 3. Oscillating function with singularity.

The values of the coefficients γ_{inm}, d_{im} and b_m, needed for the solution of the matrix equation, can be obtained analytically (see Fichte et al., 2004). The calculations resulting from the analytical approach are discussed in detail in Fichte (@).

Once the values of these coefficients are known, they can be used to express the vector potential outside the shielding bar as a series of known functions $A_k(x, y),\ k = 1, ..., 4$:

$$\begin{aligned} A_\mathrm{a}(x, y) = \sum_{n=1}^{M} v_{1n}A_1(x, y) &+ v_{2n}A_2(x, y) \\ &+ v_{3n}A_3(x, y) + v_{4n}A_4(x, y). \end{aligned}$$

This method, using analytical solutions, has been used to calculate the shielding effects of a conducting bar (Fichte et al., 2005).

4 Numerical treatment of integrals

When we insert the series representation of A_i^* into the BIE (4), we can transform the resulting integrals into expressions like

$$\int_0^a \cos(n\xi)\cdot\ln(\xi)\,\mathrm{d}\xi \quad , n \int \mathbb{N}. \tag{7}$$

Standard schemes for numerical integration do not lead to appropriate results due to the oscillatory nature of the cosine function and the singularity of the logarithmic term.

An example of an integrand is displayed in Fig. 3, with $f(x) = \ln|x - \frac{a}{2}|\cos(10\pi x)$.

In a novel approach we use numerical integration methods which deliver highly precise results. First, a decomposition of the domain of the integral is necessary. It is divided into three subdomains: the singularity is in the non-oscillating subdomain $[x_\mathrm{zl}, x_\mathrm{zh}]$. The edge points of this domain are the highest zero of f below the singularity and the lowest zero of f above it.

The remaining two subdomains $[0, x_{\mathrm{zl}}]$ and $[x_{\mathrm{zh}}, a]$ represent the non-singular oscillating parts of the integral:

$$\int_0^a \cos(n\xi)\cdot\ln(\xi)\,\mathrm{d}\xi = \underbrace{\int_0^{x_{\mathrm{zl}}} \cos(n\xi)\cdot\ln(\xi)\mathrm{d}\xi}_{I_{\mathrm{low}}} + \underbrace{\int_{x_{\mathrm{zl}}}^{x_{\mathrm{zh}}} \cos(n\xi)\cdot\ln(\xi)\mathrm{d}\xi}_{I_{\mathrm{sing}}} + \underbrace{\int_{x_{\mathrm{zl}}}^{a} \cos(n\xi)\cdot\ln(\xi)\mathrm{d}\xi}_{I_{\mathrm{hi}}}$$

Here, x_{zl} is the largest root of the integrand below the singularity and x_{zh} the smallest root above the singularity.

Note that by combination of the integral we only need three subdomains.

4.1 Treatment of the subdomain with oscillating integrand

We apply a Filon-Lobatto quadrature described in Iserles (2004) to the oscillating parts of the integral. As a first step the boundaries of the integral are mapped to the domain [0, 1]:

$$I_{\mathrm{low}} = \int_0^{x_{\mathrm{zl}}} \cos(n\xi)\cdot\ln(\xi)\mathrm{d}\xi$$
$$= x_{zl}\int_0^1 \cos(\underbrace{n x_{\mathrm{zl}}}_{\eta}\,\xi)\cdot\ln(x_{\mathrm{zl}}\xi)\mathrm{d}\xi.$$

Then we approximate the logarithmic function in the integral using N Legendre polynomials:

$$\ln(x_{\mathrm{zl}}\xi) \approx \sum_{k=1}^{N} P_k\left(\frac{\xi}{x_{\mathrm{zl}}}\right)\ln(c_k x_{\mathrm{zl}}),$$

where P_k denotes the Legendre polynomial of order k.

For the first two abscissas we choose the start- and the endpoint of the integration domain. The remaining $n-2$ abscissas c_k, $k=1,\ldots,N-1$, are the roots of $P'_{N-1}(x)$.

Thus, we can approximate the lower integral by:

$$I_{\mathrm{low}} \approx x_{\mathrm{zl}} \sum_{k=1}^{N} b_k(\eta)\ln(c_k h),$$

with known weight factors b_k:

$$b_k(\eta) = \int_0^1 P_k(\xi)\cos(\eta\xi)\mathrm{d}\xi.$$

The integral I_{hi}, which covers the part of the integral for $\xi > x_{\mathrm{z}(i+1)}$, is treated likewise.

The use of Filon-Lobatto quadrature results in a fairly low number of integration nodes neccessary for numerical quadrature of the highly oscillating integrals, since the accuracy of this method is improving with the number of zeros of the integrand.

4.2 Treatment of the subdomain containing the singularity

The subdomain of the original integral which contains the singularity is handled separately, acoording to a method presented in Ehrich et al. (1997).

First the integral is mapped to the domain [0, 1]. Then a Gauss quadrature is applied: the whole integrand is approximated by Legendre's polynomials and the zeros of Legendre's polynomial of the N-th order, c_k, are taken as abscissas. The corresponding weighting factors of the Gaussian integration are named w_k. With these expressions, I_{sing} is given by:

$$\int_0^1 \ln(x)\cos(\tilde{\eta}x)\,dx$$
$$= \sum_{k=1}^{N/2} \gamma_k(\alpha_k(\ln(x)\cos(\tilde{\eta}x))|_{x=\alpha_k}$$
$$+ \beta_k(\ln(x)\cos(\tilde{\eta}x)|_{x=\beta_k}).$$

The constants α_k, β_k and γ_k are:

$$\alpha_k = \frac{1}{2}\left[1 + \frac{F\{qc_k\}}{F\{q\}}\cdot e^{q^2-(qc_k)^2}\right],$$
$$\beta_k = \frac{\int_{qc_k}^{q} e^{-t^2}\,dt}{2F(q)} e^{q^2},$$
$$\gamma_k = \frac{w_i\,q}{2F(q)}\, e^{q^2-(qc_k)^2},$$

By using the function $F(\xi)$,

$$F(\xi) = \xi\sum_{k=0}^{\infty}\frac{(2\xi^2)^k}{(2k+1)!!};$$

the error function, as defined in Abramowitz and Stegun (1970),

$$\mathrm{ERF}(\xi) := \frac{2}{\sqrt{\pi}}\int_0^{\xi} e^{-u^2}\mathrm{d}u,$$

can be expressed as

$$\mathrm{ERF}(\xi) = \frac{2}{\sqrt{\pi}} e^{-\xi^2} F(\xi).$$

The constant q is an arbitrary real constant which has to be chosen in advance. This method for dealing with the logarithmic singularity is discussed in depth in Ehrich et al. (1997) and a value of $q=7$ has been established as an optimal value for precision and computation time.

Table 1. Integration nodes and computation time

Number of Zeros	relative precision					
	$10E-3$		$10E-7$		$10E-10$	
	k	t/s	k	t/s	k	t/s
10	14	$40E-3$	21	0.16	23	0.22
50	44	12.4	120	343	162	804
100	50	16.4	109	291	170	1001

While this method for computing singular integral has been used in the past, the combination with the Filon-Lobatto integration amounts to a new approach for the numerical quadrature of singular oscillating integrals.

5 Numerical results

The presented method has been used for the numerical approximation of an integral as appearing in Eq. 7. The results are displayed in Table 1. Here, n is the number of zeros of the integral, k is the number of polynomials used for the approximation and t is the computation time in seconds (on a 1 GHz Pentium system with 500 MB RAM).

6 Conclusion

A plane magneto-quasistatic eddy current problem has been described by a boundary integral equation. To obtain a solution to this integral equation, one has to solve a kind of integral which is highly oscillating and contains a singularity. A novel method for the approximation of those integrals has been developed. The results have been compared to analytical solutions.

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