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# On the applicability of conventional transmission line theory within cavities

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**Abstract.** We investigate whether or not conventional transmission line theory needs to be modified if transmission lines are considered that are located in a cavity rather than in free space. Our analysis is based on coupled Pocklington's equations that can be reduced to integral equations for the antenna mode and the transmission line mode. Under the usual assumptions of conventional transmission line theory these modes do approximately decouple within a cavity. As a result, cavity properties will primarily influence the antenna mode but not the transmission line mode.

## 1 Introduction

Interior problems of Electromagnetic Compatibility analysis involve electric and electronic components that are located within cavities (Tesche et al., 1997; Lee, 1995). Usually, transmission lines will constitute a part of these components. To model the electromagnetic propagation along transmission lines we have to resort to the Maxwell theory. It is desirable to simplify Maxwell's equations to Telegrapher equations since solutions of Telegrapher equations are fairly easy to obtain. But in case of interior problems we have to examine if these simplification can be made inside a cavity and this is the subject of this paper. Our strategy will be to exhibit the steps that are necessary to derive conventional transmission line theory from integral equations of antenna theory. There already is a number of such derivations (see, for example, King, 1955; Tkachenko et al., 1995; Tesche et al., 1997; Haase et al., 2004). These approaches use electric field integral equations as physical basis but differ in the assumptions and approximations that are made in order to arrive at the conventional transmission line theory. Also they assume, implicitly or explicitly, that the transmission lines are located in free space. The distinctive feature of our approach is that we take advantage of a separation of antenna mode currents and transmission line mode currents right from the beginning.

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Conventional transmission lines are metallic structures that transmit electromagnetic signals and energy. In this respect they are very similar to systems of transmitting and receiving antennas. However, the physical mechanisms that govern the electromagnetic transmission along transmission lines is quite different if compared to the electromagnetic transmission between pairs of antennas, compare Fig. 1.

In between a pair of antennas the electromagnetic transmission results from a propagating electromagnetic field which, for practical purposes, can often be approximated by a radiation field. This does not mean that in such a situation no Coulomb fields are present. Coulomb fields will be related to the electric charges that move along the antennas and constitute their near-fields. But in many cases the transmitting and receiving antennas are sufficiently far apart such that the main coupling is mediated by the radiation field which resembles a freely propagating electromagnetic field. Electric charges are not involved in the actual electromagnetic transmission that happens in between the antennas. They only are required at the beginning and at the end of the transmission in order to, respectively, generate and receive the transmitting electromagnetic field.

The electromagnetic transmission along a transmission line does involve electric charges. These charges are located on the transmission line which normally consists of a highly conducting material. They are accompanied by Coulomb fields which dominate their mutual electromagnetic interaction at short distances. While the electric charges get accelerated they will produce radiation fields. In particular, this will happen at high frequencies or if the transmission line is strongly curved or bent. Normally, such a creation of radiation fields by the electric charges on the transmission line is an unwanted effect which influences the properties of the transmission line. For many situations this influence is small

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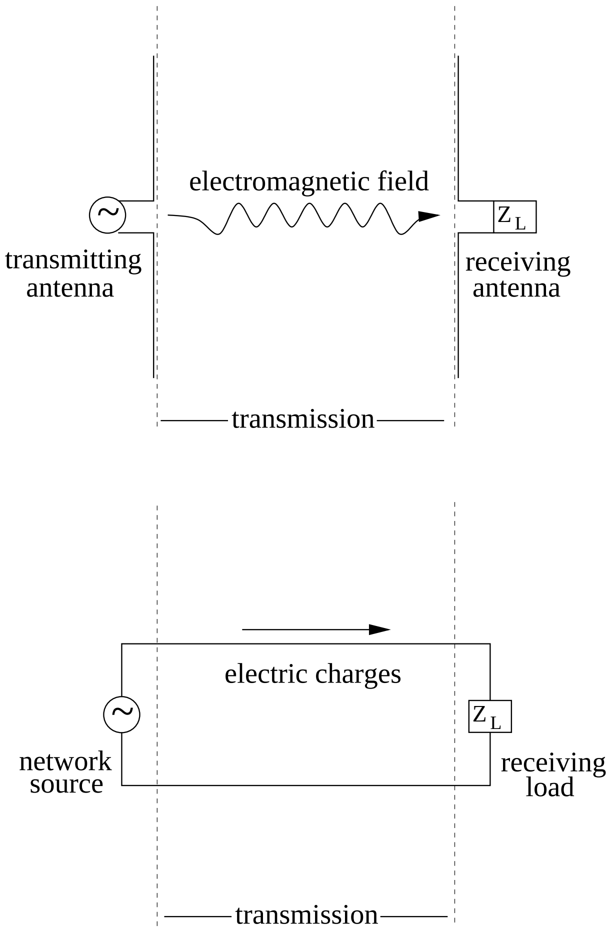


**Fig. 1.** Electromagnetic transmission by means of a pair of antennas (upper part) and a transmission line (lower part). In between the antennas an electromagnetic field mediates the actual transmission while the transmission line provides electric charges that mediate the transmission between the source and the load.

and negligible. Therefore, in the conventional transmission line theory focus is put on the electric charges and their accompanying Coulomb fields.

The conventional transmission line theory can be derived from the Maxwell theory as a limiting case and contains the electric current, representing electric charges, and the electric voltage, representing the associated Coulomb fields, as main physical quantities. Clearly, these two quantities are not independent of each other. They are related by the Telegrapher equations, which constitute a set of coupled first order differential equations, and are much easier to solve than the Maxwell equations.

In the derivation of the Telegrapher equations from the Maxwell equations it is customary to consider some, a priori, arbitrary transmission line and to assume a number of restrictions (Tesche et al., 1997):

1. The conductors are geometrically uniform, i.e. the transmission line is not curved or bent.

2. The distance between the conductors of the transmission line is small compared to the wavelength of the exciting electromagnetic field.

3. The thickness of the conductors of the transmission is small if compared to the wavelength of the exciting electromagnetic field.

4. The conductors are perfectly conducting.

The second and third of these restrictions are not clearly cut since “smallness” with respect to a wavelength is not a precise notion. The reason for these approximate criteria is that, in fact, one would like to remove the influence of radiation fields on the transmission line. However, Coulomb fields and radiation fields are inseparably intertwined. Therefore, in the conventional transmission line theory, one only takes into account electromagnetic interactions between electric charges at short distances where Coulomb fields dominate and radiation fields can be neglected. Also the first and fourth restriction are put forth to avoid an influence of radiation fields on the transmission line. In contrast to the second and third restriction they can be formulated in a mathematically exact way with no approximations involved. Since in the derivation of the Telegrapher equations approximations are inevitable it is often acceptable to relax the first and fourth conditions to some degree and allow for transmission lines which are slightly bent, i.e. which are characterized by radii of curvature that are large compared to the wavelength of the exciting field, and which are good conducting rather than perfectly conducting, i.e. which are characterized by a conductivity $\sigma$ that fulfills the requirement $|\sigma| \gg |\varepsilon\omega|$.

It has been mentioned that in the derivation of the conventional transmission line theory it usually is assumed that the transmission line is located in free space. It follows that in the derivation of the Telegrapher equations the Green’s function of free space is employed. If we want to consider a transmission line within a resonating environment we may employ a cavity’s Green’s function rather than the Green’s function of free space. Thus, it is necessary to check if this modification has an influence on the validity of the Telegrapher equations of the conventional transmission line theory.

We stress that the motivation to work within the framework of transmission line theory stems from the simplicity of the Telegrapher equations if compared to the Maxwell equations. Alternatively, we can always work on the level of electric field integral equations and directly apply approximate analytic methods or numerical methods.

## 2 Coupled Pocklington's equations, antenna and transmission line mode

We consider a set of coupled Pocklington's equations that models the electromagnetic coupling to a system of wires and represents a transmission line. For concreteness we consider two wires and assume a thin-wire approximation. Then the corresponding coupled Pocklington's equations are given by Nakano (1996)

$$j\omega\mu\left[\int_{\text{wire }1}\overline{\boldsymbol{G}}^E(\tau_1,\tau_1')\boldsymbol{I}_1(\tau_1')\,d\tau_1' + \int_{\text{wire }2}\overline{\boldsymbol{G}}^E(\tau_1,\tau_2')\boldsymbol{I}_2(\tau_2')\,d\tau_2'\right]\cdot\boldsymbol{e}_{\tau_1} = E^{\text{inc}}_{\text{tan}}(\tau_1)\,, \quad (1)$$

$$j\omega\mu\left[\int_{\text{wire }1}\overline{\boldsymbol{G}}^E(\tau_2,\tau_1')\boldsymbol{I}_1(\tau_1')\,d\tau_1' + \int_{\text{wire }2}\overline{\boldsymbol{G}}^E(\tau_2,\tau_2')\boldsymbol{I}_2(\tau_2')\,d\tau_2'\right]\cdot\boldsymbol{e}_{\tau_2} = E^{\text{inc}}_{\text{tan}}(\tau_2)\,. \quad (2)$$

Here we introduced the variables $\tau_1$, $\tau_2$ that parameterize the length of wire 1 and wire 2, respectively. Fixed values of these variables represent fixed wire positions. The unit vectors $\boldsymbol{e}_{\tau_1}$, $\boldsymbol{e}_{\tau_1}$ are tangent to the line-like wires at $\tau_1$, $\tau_2$. The currents $\boldsymbol{I}_1(\tau_1)$, $\boldsymbol{I}_2(\tau_2)$ result from the thin-wire approximation and are defined by

$$\boldsymbol{I}_i(\tau_i) := I_i\boldsymbol{e}_{\tau_i} \quad (3)$$

for $i=1,2$. The scalar $I_i$ is the value of the electric current at the wire position $\tau_i$. Finally, we denote by $\overline{\boldsymbol{G}}^E$ the dyadic Green's function for the electric field (Tai, 1994).

If the wires form a transmission line we expect that they can be parameterized by a common coordinate $\xi$ with $\xi=\xi_0$ at the beginning and $\xi=\xi_L$ at the end of the line, compare Fig. 2. We take this coordinate as a common integration variable and write Eqs. (1) and (2) as

$$j\omega\mu\left[\int_{\xi_0}^{\xi_L}\left(\overline{\boldsymbol{G}}^E(\tau_1,\tau_1')\boldsymbol{I}_1(\tau_1')\frac{\partial\tau_1'}{\partial\xi'} + \overline{\boldsymbol{G}}^E(\tau_1,\tau_2')\boldsymbol{I}_2(\tau_2')\frac{\partial\tau_1'}{\partial\xi'}\right)d\xi'\right]\cdot\boldsymbol{e}_{\tau_1} = E^{\text{inc}}_{\text{tan}}(\tau_1)\,, \quad (4)$$

$$j\omega\mu\left[\int_{\xi_0}^{\xi_L}\left(\overline{\boldsymbol{G}}^E(\tau_2,\tau_1')\boldsymbol{I}_1(\tau_1')\frac{\partial\tau_1'}{\partial\xi'} + \overline{\boldsymbol{G}}^E(\tau_2,\tau_2')\boldsymbol{I}_2(\tau_2')\frac{\partial\tau_2'}{\partial\xi'}\right)d\xi'\right]\cdot\boldsymbol{e}_{\tau_2} = E^{\text{inc}}_{\text{tan}}(\tau_2)\,. \quad (5)$$

The variables $\tau_1$, $\tau_2$ are now understood as functions of the parameter $\xi$.

Next we introduce two currents $\boldsymbol{I}_{\text{A}}$ and $\boldsymbol{I}_{\text{TL}}$ as linear combinations of $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$,

$$\boldsymbol{I}_{\text{A}} := \tfrac{1}{2}\left(\boldsymbol{I}_1+\boldsymbol{I}_2\right)\,, \quad (6)$$

$$\boldsymbol{I}_{\text{TL}} := \tfrac{1}{2}\left(\boldsymbol{I}_1-\boldsymbol{I}_2\right)\,. \quad (7)$$

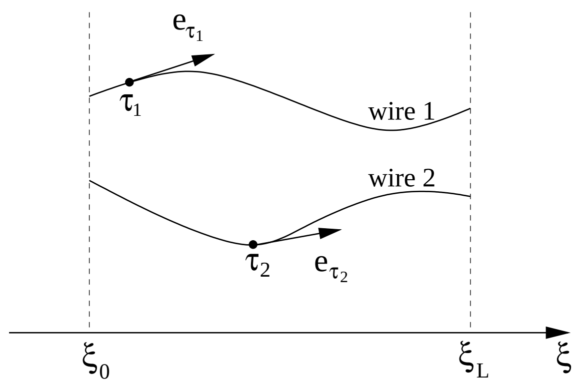


**Fig. 2.** Introduction of a common variable $\xi$ which parameterizes the wires of a transmission line.

The inverse equations are

$$\boldsymbol{I}_1 = \boldsymbol{I}_{\text{A}} + \boldsymbol{I}_{\text{TL}}\,, \quad (8)$$

$$\boldsymbol{I}_2 = \boldsymbol{I}_{\text{A}} - \boldsymbol{I}_{\text{TL}}\,. \quad (9)$$

These identifications are well-known from the conventional transmission line theory where $\boldsymbol{I}_{\text{A}}$ represents the so-called "antenna mode" or "common mode" and $\boldsymbol{I}_{\text{TL}}$ represents the so-called "tranmission line mode" or "differential mode". In our present context these identifications are still formal. We note that neither $\boldsymbol{I}_{\text{A}}$ nor $\boldsymbol{I}_{\text{TL}}$ need to be tangent to one of the wires. But it is clear that we still may split $\boldsymbol{I}_{\text{A}}$ and $\boldsymbol{I}_{\text{TL}}$ into a component and a unit vector,

$$\boldsymbol{I}_{\text{A}} = \frac{1}{2}\left(I_1\boldsymbol{e}_{\tau_1}+I_2\boldsymbol{e}_{\tau_2}\right) =: I_{\text{A}}\boldsymbol{e}_{I_A}\,, \quad (10)$$

$$\boldsymbol{I}_{\text{TL}} = \frac{1}{2}\left(I_1\boldsymbol{e}_{\tau_1}-I_2\boldsymbol{e}_{\tau_2}\right) =: I_{\text{TL}}\boldsymbol{e}_{I_{TL}}\,. \quad (11)$$

If the relations (8) and (9) are inserted into Eqs. (4) and (5) it is simple to find

$$j\omega\mu\left[\int_{\xi_0}^{\xi_L}\left(\left[\overline{\boldsymbol{G}}^E(\tau_1,\tau_1')\frac{\partial\tau_1'}{\partial\xi'}+\overline{\boldsymbol{G}}^E(\tau_1,\tau_2')\frac{\partial\tau_2'}{\partial\xi'}\right]\boldsymbol{I}_{\text{A}}(\xi') + \left[\overline{\boldsymbol{G}}^E(\tau_1,\tau_1')\frac{\partial\tau_1'}{\partial\xi'}-\overline{\boldsymbol{G}}^E(\tau_1,\tau_2')\frac{\partial\tau_2'}{\partial\xi'}\right]\boldsymbol{I}_{\text{TL}}(\xi')\right)d\xi'\right]\cdot\boldsymbol{e}_{\tau_1} = E^{\text{inc}}_{\text{tan}}(\tau_1)\,, \quad (12)$$

$$j\omega\mu\left[\int_{\xi_0}^{\xi_L}\left(\left[\overline{\boldsymbol{G}}^E(\tau_2,\tau_1')\frac{\partial\tau_1'}{\partial\xi'}+\overline{\boldsymbol{G}}^E(\tau_2,\tau_2')\frac{\partial\tau_2'}{\partial\xi'}\right]\boldsymbol{I}_{\text{A}}(\xi') + \left[\overline{\boldsymbol{G}}^E(\tau_2,\tau_1')\frac{\partial\tau_1'}{\partial\xi'}-\overline{\boldsymbol{G}}^E(\tau_2,\tau_2')\frac{\partial\tau_2'}{\partial\xi'}\right]\boldsymbol{I}_{\text{TL}}(\xi')\right)d\xi'\right]\cdot\boldsymbol{e}_{\tau_2} = E^{\text{inc}}_{\text{tan}}(\tau_2)\,. \quad (13)$$

We both add and subtract these equations and obtain

$$
\begin{aligned}
j\omega\mu \int_{\xi_0}^{\xi_L} \Big( & G^E_{+\mathrm{A}}(\tau_1, \tau_2, \tau_1', \tau_2') I_{\mathrm{A}}(\xi') \\
& + G^E_{+\mathrm{TL}}(\tau_1, \tau_2, \tau_1', \tau_2') I_{\mathrm{TL}}(\xi') \Big)\, d\xi' \\
& = E^{\mathrm{inc}}_{\tan}(\tau_1) + E^{\mathrm{inc}}_{\tan}(\tau_2)\,, \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
j\omega\mu \int_{\xi_0}^{\xi_L} \Big( & G^E_{-\mathrm{A}}(\tau_1, \tau_2, \tau_1', \tau_2') I_{\mathrm{A}}(\xi') \\
& + G^E_{-\mathrm{TL}}(\tau_1, \tau_2, \tau_1', \tau_2') I_{\mathrm{TL}}(\xi') \Big)\, d\xi' \\
& = E^{\mathrm{inc}}_{\tan}(\tau_1) - E^{\mathrm{inc}}_{\tan}(\tau_2)\,, \qquad (15)
\end{aligned}
$$

where we introduced the abbreviations

$$
\begin{aligned}
& G^E_{+\mathrm{A}}(\tau_1, \tau_2, \tau_1', \tau_2') := \qquad (16) \\
& \quad \boldsymbol{e}_{\tau_1} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_1, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} + \overline{\boldsymbol{G}}^E(\tau_1, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_A} \\
& \quad + \boldsymbol{e}_{\tau_2} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_2, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} + \overline{\boldsymbol{G}}^E(\tau_2, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_A}\,,
\end{aligned}
$$

$$
\begin{aligned}
& G^E_{+\mathrm{TL}}(\tau_1, \tau_2, \tau_1', \tau_2') := \qquad (17) \\
& \quad \boldsymbol{e}_{\tau_1} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_1, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} - \overline{\boldsymbol{G}}^E(\tau_1, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_T L} \\
& \quad + \boldsymbol{e}_{\tau_2} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_2, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} - \overline{\boldsymbol{G}}^E(\tau_2, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_T L}\,,
\end{aligned}
$$

$$
\begin{aligned}
& G^E_{-\mathrm{A}}(\tau_1, \tau_2, \tau_1', \tau_2') := \qquad (18) \\
& \quad \boldsymbol{e}_{\tau_1} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_1, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} + \overline{\boldsymbol{G}}^E(\tau_1, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_A} \\
& \quad - \boldsymbol{e}_{\tau_2} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_2, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} + \overline{\boldsymbol{G}}^E(\tau_2, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_A}\,,
\end{aligned}
$$

$$
\begin{aligned}
& G^E_{-\mathrm{TL}}(\tau_1, \tau_2, \tau_1', \tau_2') := \qquad (19) \\
& \quad \boldsymbol{e}_{\tau_1} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_1, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} - \overline{\boldsymbol{G}}^E(\tau_1, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_T L} \\
& \quad - \boldsymbol{e}_{\tau_2} \cdot \left( \overline{\boldsymbol{G}}^E(\tau_2, \tau_1') \frac{\partial \tau_1'}{\partial \xi'} - \overline{\boldsymbol{G}}^E(\tau_2, \tau_2') \frac{\partial \tau_2'}{\partial \xi'} \right) \cdot \boldsymbol{e}_{I_T L}\,.
\end{aligned}
$$

## 3 Decoupling of antenna and transmission line mode in free space

The expressions we obtained so far look more complicated than the original Eqs. (1) and (2) that we started from. To nevertheless appreciate this form of the coupled Pocklington's equations we specialize to the case of straight and parallel wires. Then we may align a Cartesian coordinate system such that the $z$-axis is parallel to the wires and may choose $\xi = z$. This leads to the simplifications

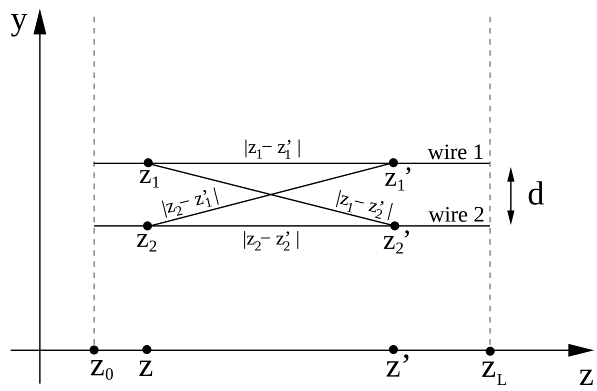


**Fig. 3.** Geometry of a straight two-wire transmission line.

$$
\boldsymbol{e}_{\tau_1} = \boldsymbol{e}_{\tau_2} = \boldsymbol{e}_{I_A} = \boldsymbol{e}_{I_T L}\,, \qquad (20)
$$

$$
\boldsymbol{e}_{\tau_{1,2}} \cdot \overline{\boldsymbol{G}}^E \cdot \boldsymbol{e}_{I_{\mathrm{A,TL}}} = G^E_{zz}\,, \qquad (21)
$$

$$
\frac{\partial \tau_1}{\partial \xi} = \frac{\partial \tau_2}{\partial \xi} = 1\,. \qquad (22)
$$

Accordingly, Eqs. (16) – (19) reduce to

$$
\begin{aligned}
G^E_{+\mathrm{A}}(z_1, z_2, z_1', z_2') = {} & G^E_{zz}(z_1, z_1') + G^E_{zz}(z_1, z_2') \\
& + G^E_{zz}(z_2, z_1') + G^E_{zz}(z_2, z_2')\,, \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
G^E_{+\mathrm{TL}}(z_1, z_2, z_1', z_2') = {} & G^E_{zz}(z_1, z_1') - G^E_{zz}(z_1, z_2') \\
& + G^E_{zz}(z_2, z_1') - G^E_{zz}(z_2, z_2')\,, \qquad (24)
\end{aligned}
$$

$$
\begin{aligned}
G^E_{-\mathrm{A}}(z_1, z_2, z_1', z_2') = {} & G^E_{zz}(z_1, z_1') + G^E_{zz}(z_1, z_2') \\
& - G^E_{zz}(z_2, z_1') - G^E_{zz}(z_2, z_2')\,, \qquad (25)
\end{aligned}
$$

$$
\begin{aligned}
G^E_{-\mathrm{TL}}(z_1, z_2, z_1', z_2') = {} & G^E_{zz}(z_1, z_1') - G^E_{zz}(z_1, z_2') \\
& - G^E_{zz}(z_2, z_1') + G^E_{zz}(z_2, z_2')\,, \qquad (26)
\end{aligned}
$$

Let us now suppose that the transmission line is located in free space such that the Green's function is that of free space, $G^E_{zz} = G^E_{\mathrm{free}\ zz}$. It is, in particular, translation invariant,

$$
G^E_{\mathrm{free}\ zz}(z, z') = G^E_{\mathrm{free}\ zz}(|z - z'|)\,. \qquad (27)
$$

For straight, parallel wires we have, compare Fig. 3,

$$
\begin{aligned}
& |z_1 - z_1'| = |z_2 - z_2'| \qquad (28) \\
& \quad \Longrightarrow \quad G^E_{\mathrm{free}\ zz}(z_1, z_1') = G^E_{\mathrm{free}\ zz}(z_2, z_2')\,, \\
& |z_1 - z_2'| = |z_2 - z_1'| \qquad (29) \\
& \quad \Longrightarrow \quad G^E_{\mathrm{free}\ zz}(z_1, z_2') = G^E_{\mathrm{free}\ zz}(z_2, z_1')\,.
\end{aligned}
$$

It follows that the relations (23) – (26) reduce to

$$G^E_{+\mathrm{A}}(z_1, z_2, z_1', z_2') = 2\left(G^E_{\mathrm{free}\ zz}(z_1, z_1') + G^E_{\mathrm{free}\ zz}(z_1, z_2')\right), \tag{30}$$

$$G^E_{+\mathrm{TL}}(z_1, z_2, z_1', z_2') = 0\,, \tag{31}$$

$$G^E_{-\mathrm{A}}\ (z_1, z_2, z_1', z_2') = 0\,, \tag{32}$$

$$G^E_{-\mathrm{TL}}(z_1, z_2, z_1', z_2') = 2\left(G^E_{\mathrm{free}\ zz}(z_1, z_1') - G^E_{\mathrm{free}\ zz}(z_1, z_2')\right), \tag{33}$$

and in view of the integral equation system (14), (15) it is recognized that the antenna mode $I_\mathrm{A}$ and the transmission line mode $I_\mathrm{TL}$ completely decouple,

$$j\omega\mu \int_{z_0}^{z_L} G^E_{+\mathrm{A}}(z_1, z_2, z_1', z_2') I_\mathrm{A}(z')\, dz' = +E^{\mathrm{inc}}_{\tan}(z_1) + E^{\mathrm{inc}}_{\tan}(z_2)\,, \tag{34}$$

$$j\omega\mu \int_{z_0}^{z_L} G^E_{-\mathrm{TL}}(z_1, z_2, z_1', z_2') I_\mathrm{TL}(z')\, dz' = +E^{\mathrm{inc}}_{\tan}(z_1) - E^{\mathrm{inc}}_{\tan}(z_2)\,. \tag{35}$$

## 4 Reduction to conventional transmission line theory in free space

The antenna mode current $I_\mathrm{A}$ in Eq. (34) vanishes at the beginning and at the end of the transmission line. This is analogous to the boundary conditions of the antenna current on a single wire antenna. Also the Green's function $G^E_{+\mathrm{A}}(z_1, z_2, z_1', z_2')$ is similar to the kernel of the Pocklington's equation for a single wire antenna since in Eq. (30) the terms $G^E_{\mathrm{free}\ zz}(z_1, z_1')$ and $G^E_{\mathrm{free}\ zz}(z_1, z_2')$ add up and only considerably differ if the distance $|z - z'|$ is smaller or of the order of the distance $d$ between wire 1 and wire 2. It follows that Eq. (34) can be solved with methods of antenna theory in cavities (Gronwald, 2005).

The conventional transmission line theory is contained in Eq. (35). To explicitly see this we first note that a Pocklington's equation

$$j\omega\mu \int_{z_L}^{z_o} G^E_{zz}(z, z') I(z')\, dz' = E^{\mathrm{inc}}_{\tan}(z) \tag{36}$$

is equivalent to a mixed potential integral equation (Nakano, 1996)

$$\frac{1}{j\omega\varepsilon} \int_{z_0}^{z_L} \left[ \frac{\partial G^\phi(z, z')}{\partial z} \frac{\partial I(z')}{\partial z'} + k^2 G^A_{zz}(z, z') I(z') \right] dz' = -E^{\mathrm{inc}}_z(z) \tag{37}$$

with $G^\phi(z, z')$ and $G^A_{zz}(z, z')$ indicating the Green's functions for the scalar potential $\phi$ and the magnetic vector potential $\boldsymbol{A}$ in the Lorenz gauge, respectively. We introduce a per-unit-length charge $q'$ by the continuity equation

$$\frac{\partial I}{\partial z} + j\omega q' = 0 \tag{38}$$

and define a potential $V^{q'}$ by

$$V^{q'} := \frac{1}{\varepsilon} \int_{z_0}^{z_L} G^\phi(z, z') q'(z')\, dz'\,. \tag{39}$$

Furthermore, in view of Eq. (33), we introduce the combinations

$$G^\phi_{-\mathrm{TL}}(z, z') = 2\left(G^\phi_{\mathrm{free}}(z_1, z_1') - G^\phi_{\mathrm{free}}(z_1, z_2')\right), \tag{40}$$

$$G^A_{-\mathrm{TL}}(z, z') = 2\left(G^A_{\mathrm{free}\ zz}(z_1, z_1') - G^A_{\mathrm{free}\ zz}(z_1, z_2')\right), \tag{41}$$

and note that $G^\phi_{-\mathrm{TL}}$ and $G^A_{-\mathrm{TL}}$ are localized functions that are characterized by a sharp peak in the domain where the distance $|z - z'|$ is small. This feature is often used in the derivation of the conventional transmission line theory (see Tkachenko et al., 1995, for example). It leads to the simplifications

$$\int_{z_0}^{z_L} G^\phi_{-\mathrm{TL}}(z, z') q'(z')\, dz' \approx q'(z) \int_{z_0}^{z_L} G^\phi_{-\mathrm{TL}}(z, z')\, dz' \tag{42}$$

$$\int_{z_0}^{z_L} G^A_{-\mathrm{TL}}(z, z') I(z')\, dz' \approx I(z) \int_{z_0}^{z_L} G^\phi_{-\mathrm{TL}}(z, z')\, dz' \tag{43}$$

The mixed potential integral Eq. (37) and the continuity Eq. (38) can now be written as conventional Telegrapher equations

$$\frac{\partial V^{q'}}{\partial z}(z) + j\omega L' I(z) = -\left(E^{\mathrm{inc}}_{\tan}(z_1) - E^{\mathrm{inc}}_{\tan}(z_2)\right), \tag{44}$$

$$\frac{\partial I}{\partial z}(z) + j\omega C' V^{q'}(z) = 0\,, \tag{45}$$

with

$$C' := \frac{\varepsilon}{\int_{z_0}^{z_L} G^\phi_{-\mathrm{TL}}(z, z')\, dz'} \approx \frac{\pi\varepsilon}{\ln(d/\rho)}\,, \tag{46}$$

$$L' := \mu \int_{z_0}^{z_L} G^A_{-\mathrm{TL}}(z, z')\, dz' \approx \frac{\mu}{\pi} \ln(d/\rho)\,. \tag{47}$$

Here the distance between the wires is, as before, denoted by $d$ and the wire radius is denoted by $\rho$.

## 5 Transmission lines in cavities

In the derivation of the Telegrapher Eqs. (44) and (45) from the Pocklington's Eqs. (1) and (2) we employed three times the properties of free space Green's functions:

1. To decouple for straight, parallel wires the antenna mode from the transmission line mode we used translation invariance of the free space Green's function $G^E_{\mathrm{free}\;zz}$ in Eq. (27).

2. To pull in Eqs. (42) and (43) the charge density $q'$ and the current $I$ out of the integrals we used the strong Coulomb singularity that is contained in the free space Green's functions $G^{\phi}_{\mathrm{free}}$ and $G^A_{\mathrm{free}\;zz}$.

3. To calculate the per-unit-length parameter $L'$ and $C'$ we used the explicit mathematical expression for the free space Green's functions $G^{\phi}_{\mathrm{free}}$ and $G^A_{\mathrm{free}\;zz}$.

Within a cavity the Green's functions can always be written as a sum of a free space part and a boundary part which takes into account the effect of the cavity walls,

$$G_{\mathrm{cav}} = G_{\mathrm{free}} + \tilde{G}\,. \tag{48}$$

We now state that the steps 1.–3. can approximately be performed within a cavity:
1. The decoupling of antenna and transmission line mode requires the kernels $G^E_{+\mathrm{TL}}(z_1,z_2,z_1',z_2')$ and $G^E_{-\mathrm{A}}(z_1,z_2,z_1',z_2')$ to vanish. From the relations (24) and (25), and Fig. 3 with $d=y_1-y_2$ it follows that it is meaningful to consider the Taylor expansions

$$\tilde{G}^E_{zz}(z_1,z_2') \approx \tilde{G}^E_{zz}(z_1,z_1') - \tfrac{\partial \tilde{G}^E_{zz}}{\partial y}(z_1,z_1')\,d \tag{49}$$

$$\tilde{G}^E_{zz}(z_1,z_2') \approx \tilde{G}^E_{zz}(z_2,z_2') + \tfrac{\partial \tilde{G}^E_{zz}}{\partial y}(z_2,z_2')\,d \tag{50}$$

$$\tilde{G}^E_{zz}(z_2,z_1') \approx \tilde{G}^E_{zz}(z_2,z_2') + \tfrac{\partial \tilde{G}^E_{zz}}{\partial y}(z_2,z_2')\,d \tag{51}$$

$$\tilde{G}^E_{zz}(z_2,z_1') \approx \tilde{G}^E_{zz}(z_1,z_1') - \tfrac{\partial \tilde{G}^E_{zz}}{\partial y}(z_1,z_1')\,d \tag{52}$$

since then

$$\begin{aligned}\tilde{G}^E_{+\mathrm{TL}}(z_1,z_2,z_1',z_2') &= \tilde{G}^E_{-\mathrm{A}}(z_1,z_2,z_1',z_2') \\ &\approx \left(\frac{\partial \tilde{G}^E_{zz}}{\partial y}(z_1,z_1') + \frac{\partial \tilde{G}^E_{zz}}{\partial y}(z_2,z_2')\right) d\,. \end{aligned} \tag{53}$$

If the terms involving the derivative of the Green's function are small the kernels $G^E_{+\mathrm{TL}}(z_1,z_2,z_1',z_2')$ and $G^E_{-\mathrm{A}}(z_1,z_2,z_1',z_2')$ are small as well and, as a result, antenna and transmission line mode approximately decouple. For a general discussion of this point one might represent the cavities Green's function $G_{\mathrm{cav}}$ by means of an expansion in rotational and irrotational eigenvectors (Tai, 1994).

The rotational eigenvectors are solutions of sourceless Helmholtz equations and their spatial variation is of the order of the wavelength considered. Then the contributions of these rotational eigenvectors to the first order terms in Eqs. (49)–(52) will be of the order of $kd$. Thus, according to the usual assumption $kd \ll 1$ of conventional transmission line theory, these contributions will be small.

The irrotational eigenvectors are solutions of electrostatic Poisson equations and contain the Coulomb singularity. Their spatial variation diverges close to a Coulomb singularity and, otherwise, decays quickly. Since $\tilde{G} = G_{\mathrm{cav}} - G_{\mathrm{free}}$ contains no Coulomb singularity we expect that, in general, the irrotational eigenvectors will lead to no significant spatial variation of $\tilde{G}$. There is only one situation where this argument does not apply and this is when the distance of one of the wires to a cavity wall is of the order or smaller than the wire distance $d$. In this case there can be a dominant Coulomb interaction with the cavity wall (that is, with the mirrored wires) which is embedded in $\tilde{G}$ since $G_{\mathrm{free}}$ does not take into account scattering contributions from the cavity walls. Then the spatial variation of $\tilde{G}$ might become large and one would need to actually calculate the derivatives of Eq. (53) for the specific configuration in order to see if they still lead to only small corrections.
2. We need to reconsider the approximations (42) and (43) with

$$G^{\phi}_{-\mathrm{TL}}(z,z') = 2\big(G^{\phi}_{\mathrm{cav}}(z_1,z_1') - G^{\phi}_{\mathrm{cav}}(z_1,z_2')\big)\,, \tag{54}$$

$$G^{A}_{-\mathrm{TL}}(z,z') = 2\big(G^{A}_{\mathrm{cav}\,zz}(z_1,z_1') - G^{A}_{\mathrm{cav}\,zz}(z_1,z_2')\big)\,. \tag{55}$$

The approximations (42) and (43) are valid since the differences of the form $G(z_1,z_1') - G(z_1,z_2')$ approximately cancel if $|z-z'|$ is much larger than the separation $d$. The sharp peak of $G^{\phi}_{-\mathrm{TL}}(z,z')$ and $G^{A}_{-\mathrm{TL}}(z,z')$ at $z=z'$ is due to the fact that for $z\to z'$ we have, with $\rho \ll d$,

$$G^{\phi}_{-\mathrm{TL}}(z,z') = G^{A}_{-\mathrm{TL}}(z,z') \approx \frac{1}{2\pi}\left(\frac{1}{\rho} - \frac{1}{d}\right) \gg 1\,. \tag{56}$$

This feature is unaffected by the presence of the cavity. Rotational contributions of $\tilde{G}$ to $G_{\mathrm{cav}}$ will approximately cancel within the differences (54) and (55), and Coulomb interactions with the cavity walls will not significantly change the property (56).
3. The electrostatic calculation that led to the values of the per-unit-length parameters (47) and (46) consists of the evaluation of the integrals $\int_{-L/2}^{L/2} G^{\phi,A}_{-\mathrm{TL}}(z,z')\,dz'$. Within a cavity these parameters will significantly change only if one or both of the wires is close to a cavity wall such that the Coulomb field in the vicinity of the transmission lines is significantly perturbed by the presence of the cavity. In this case the parameters $L'$, $C'$ have to be calculated from an electrostatic calculation which takes into account the cavity wall.

## 6 Conclusions

Conventional transmission line is applicable not only in free space but also within a cavity. The dynamical (rotational) electromagnetic fields within the cavity will mainly couple to the antenna mode current which approximately is decoupled from the transmission line mode. In particular, resonances of the cavity will strongly interact with the antenna mode but not with the transmission line mode. Some care must be taken if the presence of the cavity has a significant influence on the (irrotational) Coulomb fields in the vicinity of the transmission line. In this case, quantitative changes of the per-unit-length parameters are expected and need to be calculated from the actual transmission line configuration.

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